Equilibrium structure of a bidimensional asymmetric city

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Introduction

Firms and individuals compete for land use. Structure of cities: way land is shared between those uses in terms of densities.

Competitive equilibrium models where structure results from rational behaviour : Fujita and Ogawa (1980, 1982), Fujita-Smith (1987), Fujita (1989), Lucas and Rossi-Hansberg (2002).

Existence of an equilibrium but these are one-dimensional models. Our main departure from the Lucas-Rossi-Hansberg model : monetary cost as in Berliant et al. (2002).

driving forces for concentration: production externalities, transportation costs,

driving forces for dispersion: agents value space,

constraints: land market, rents.

Plan of the talk

① Model

② Rational behavior

③ Definition of equilibria

④ Optimal transportation

⑤ Existence of equilibiria: sketch of proof

⑥ Concluding remarks

The model

The city: Ω, bounded domain of $\mathbb{R}^2$. Three kinds of actors: agents, firms and landowners. A single good is consumed and produced in Ω.

Agents: identical, utility $U(c, S)$, c consumption, S surface, strictly concave, increasing in each argument,

Firms: identical, production $f(z, n)$, z productivity, n employment, continuous strictly concave in n, increasing in each argument,

Landowners: no role (absentee landlords) except they extract all the surplus.

Production externalities:

Given employment density $\nu(y)dy$ in the city, the productivity function is:

$$z(x) = Z_\nu(x) := \chi(\int_\Omega \rho(x,y)\nu(y)dy) \text{ for all} x \in \Omega \qquad (1)$$

With ρ a continuous positive kernel and χ a continuous increasing function such that $\chi(\mathbb{R}_+) \subset [\underline{z}, \overline{z}] \subset (0, +\infty)$.

Open city model: population size is not fixed (but the utility of agents is).

Agents

At equilibrium all agents have the same utility $\overline{u}$. If available revenue at $x \in \Omega$ is $\varphi = \varphi(x)$, and denoting Q the rent, one gets:

$$\varphi = V(Q) := \min\{c + QS : U(c, S) \geq \overline{u}\} \tag{2}$$

Using $Q = V^{-1}(\varphi)$ one gets $c(\varphi)$ and $S(\varphi)$.

Number of residents per unit of surface used for residential use:

$$N(\varphi) = \frac{1}{S(\varphi)}$$

note that $Q(\varphi)$ is the rent for residential use.

Firms

If, at $y \in \Omega$, productivity is z and wage is ψ the firm solves

$$q(z, \psi) := \max_{n \geq 0} f(z, n) - \psi \cdot n \qquad (3)$$

$q(z, \psi)$ is then the rent for business use. Employment $n(z, \psi)$: the solution of (3).

Landowners

At $x \in \Omega$, if productivity is z, wage is ψ and residents is φ : two rents $q(z, \psi)$ (business) and $Q(\varphi)$ (residence). Landowners determine the fraction of surface devoted to business use.

Consider two cases:

Land is allocated to the highest bidder

$$q(z(x), \psi(x)) > Q(\varphi(x)) \Rightarrow \theta(x) = 1, \tag{4}$$

$$q(z(x), \psi(x)) < Q(\varphi(x)) \Rightarrow \theta(x) = 0, \tag{5}$$

Zoning restrictions

Rules out purely business or purely residentials areas and discontinuities. Landowners' program:

$$\max_{\theta \in [\underline{\theta}, \overline{\theta}]} \theta q(z, \psi) + (1 - \theta) Q(\varphi) - g(\theta). \tag{6}$$

With $1 > \overline{\theta} > \underline{\theta} > 0$ and g strictly convex increasing. Denote by $\theta(z, \psi, \varphi)$ the solution of (6).

Remark When $\underline{\theta}$ and g "small" and $\overline{\theta}$ close to 1: continuous approximation of the (discontinuous) highest bidder case .

Densities

Zoning case

Density of residents

$$\widetilde{\mu}(z,\psi,\varphi) := (1-\theta(z,\psi,\varphi))N(\varphi) \tag{7}$$

Density of employment

$$\widetilde{\nu}(z,\psi,\varphi) := \theta(z,\psi,\varphi)n(z,\psi) \tag{8}$$

Free mobility of labor

monetary commuting cost $c(x, y)$, residents maximize wage net of commuting. Conjugacy relations between wage $\psi(.)$ and revenue $\varphi(.)$:

$$\varphi(x) = \sup_{y \in \Omega} \psi(y) - c(x, y), \; \forall x \in \Omega \tag{9}$$

$$\psi(y) = \inf_{x \in \Omega} \varphi(x) + c(x, y), \; \forall y \in \Omega \tag{10}$$

Transportation plans

$\gamma(A \times B)$ = number of agents living in A and working in B. If (μ, ν) are the densities of residents and employment (at equilibrium, they must have the same total mass), obviously, μ ands ν are the marginals of γ (notation: $\gamma \in \Pi(\mu, \nu)$). Besides, an individual living at x chooses is job location in

$$\text{argmax}_y\{\psi(y) - c(x, y)\}. \tag{11}$$

Similarly, a firm located at y hires workers from:

$$\text{argmin}_x\{\varphi(x) + c(x, y)\}. \tag{12}$$

In view of (9) and (10), this means:

$$\psi(y) - \varphi(x) = c(x, y) \ \gamma\text{-a.e.} \tag{13}$$

Equilibrium

$(\mu, \nu, \psi, \varphi)$ continuous and > 0 on Ω, and $\gamma \in \Pi(\mu, \nu)$ such that

1. $\int_\Omega \mu = \int_\Omega \nu$,
2. for all $x \in \Omega$:

$$\mu(x) = \widetilde{\mu}(Z_\nu(x), \psi(x), \varphi(x)), \text{ and } \nu(x) = \widetilde{\nu}(Z_\nu(x), \psi(x), \varphi(x)),$$

3. (ψ, φ) satisfies the conjugacy relations (9), and (10),
4. for γ-almost every $(x, y) \in \Omega \times \Omega$:

$$\psi(y) - \varphi(x) = c(x, y).$$

Pure equilibria

Equilibria such that agents with the same address do the same thing. Definition is the same as before except that the commuting plan γ is supported by the graph of a commuting map s (given x the conditional probability of job location is then $\delta_{s(x)}$).

- $s(x)$ is the job location of agents living at x,
- s is a measure preserving map between μ and ν.

Optimal transportation

Given two nonnegative measures μ and ν with the same total mass, requirements 3 and 4 exactly mean that γ solves the Monge-Kantorovich problem:

$$(\mathcal{M}_{\mu,\nu}) \quad \inf\left\{\int_{\Omega\times\Omega} c(x,y)d\gamma(x,y) \,:\, \gamma\in\Pi(\mu,\nu)\right\} \tag{14}$$

and that (ψ,φ) solve its dual:

$$(\mathcal{D}_{\mu,\nu}) \quad \sup_{\psi,\varphi}\left\{\int\psi d\nu - \int\varphi d\mu \,:\, \psi(y)-\varphi(x)\leq c(x,y),\ (x,y)\in\Omega^2\right\}.$$

Under additional conditions, optimal plans are supported by graphs of transport maps (McCann-Gangbo).

Assumptions

For the sake of simplicity (in this talk), we assume that Ω is either smooth or convex, that the cost is of the form:

$$c(x,y) = |x-y|^{\eta_0}.$$

with $\eta_0 > 0$. For the sake of simplicity again, we make the following Cobb-Douglas specifications:

$$\begin{aligned} f(z,n) &= z^{\gamma_0} n^{\alpha_0}, \\ U(c,S) &= c^{\beta_0} S^{1-\beta_0} \end{aligned}$$

with $\gamma_0 > 0$, $\beta_0 \in (0,1)$ and $\alpha_0 \in (0,1)$.

Explicit computations yield then:

$$n(z,\psi) = \left(\frac{\alpha_0 z^{\gamma_0}}{\psi}\right)^{\frac{1}{1-\alpha_0}}, \tag{15}$$

$$N(\varphi) = \beta_0^{\beta_0/(1-\beta_0)}\overline{u}^{-1/(1-\beta_0)}\varphi^{\beta_0/(1-\beta_0)} \tag{16}$$

and

$$q(z,\psi) = (1-\alpha_0)z^{\gamma_0/(1-\alpha_0)}\left(\frac{\alpha_0}{\psi}\right)^{\frac{\alpha_0}{1-\alpha_0}}, \tag{17}$$

$$Q(\varphi) = (1-\beta_0)\left(\frac{\beta_0^{\beta_0}\varphi}{\overline{u}}\right)^{1/(1-\beta_0)} \tag{18}$$

Existence

Under the assumptions above, we then have:

Theorem 1 1. ***strictly convex case:*** *if* $\eta_0 > 1$ *and* $\alpha_0 \geq 1/2$ *then there exists at least one equilibrium and every equilibrium is pure,*

2. ***sublinear case:*** *if* $0 < \eta_0 \leq 1$ *and* $\eta_0 \geq 2(1 - \alpha_0)$, *then there exists at least one equilibrium.*

Sketch of proof: start with densities (μ, ν) with same positive total mass

Step 1:

$z := Z_\nu$, determine wages and revenues (ψ, φ) conjugate by solving $(\mathcal{D}_{\mu,\nu})$.

Step 2:

Determine a constant λ such that

$$\int_\Omega \widetilde{\mu}(Z_\nu(x), \psi(x)+\lambda, \varphi(x)+\lambda)dx = \int_\Omega \widetilde{\nu}(Z_\nu(x), \psi(x)+\lambda, \varphi(x)+\lambda)dx$$

finally set:

$$T(\mu,\nu) := (\widetilde{\mu}(Z_\nu, \psi+\lambda, \varphi+\lambda), \widetilde{\nu}(Z_\nu, \psi+\lambda, \varphi+\lambda)).$$

equilibria are associated to fixed-points of T and one establishes the existence of such fixed-points by using Schauder's Theorem.

Variants and extensions

We may use the same method to prove existence of equilibria in the following cases:

- no zoning restriction: land is allocated to the highest bidder (proceed by approximation),
- more general utilities and production functions,
- more general externalities.

Concluding remarks

- to our knowledge this is the first existence result in dimension 2,
- if the problem is radially symmetric (as in Lucas-Rossi-Hansberg), there exists symmetric (radial) equilibria, are there nonsymmetric ones ?
- on costs : if costs are convex transporation plans are carried by the graph of a transport map,
- externalities (and the fact that the boundary of the city is given) imply that equilibrium necessarily involves commuting.

Open questions and perspectives

- uniqueness, comparative statics, population size at equilibrium,
- qualitative properties (polycentric vs monocentric...),
- numerical methods,
- welfare analysis,
- endogenous city shape.