A categorical invariant of flow equivalence of shifts

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(Joint work with Benjamin Steinberg, City College of New York)

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The syntactic semigroup

Let \( L \) be a language of \( A^+ \). The syntactic semigroup of \( L \) is the quotient of \( A^+ \) by the congruence

\[
\begin{align*}
  u \equiv_L v & \iff C(u) = C(v) \\
  C(w) & = \{(z, t) \in A^* \mid zw t \in L\}.
\end{align*}
\]

\[A^+/\equiv_L:\]

\[
\begin{array}{|c|c|c|}
  \hline
  & & \\
  \hline
  \hline
  \end{array}
\]
Let $\mathcal{X}$ be a subshift of $A^\mathbb{Z}$. If $\mathcal{X}$ is strictly contained in $A^\mathbb{Z}$, then the syntactic semigroup of $L(\mathcal{X})$, as a subset of $A^+$, is a semigroup with a zero.

The elements of $A^+ \setminus L$ form a class, which is the zero element.

To avoid treating $A^\mathbb{Z}$ differently, we add a zero to the syntactic semigroup of $L(A^\mathbb{Z})$.

Notation for the syntactic semigroup (with zero): $S(\mathcal{X})$. 
The syntactic semigroup

The transition semigroup of the Fischer cover of an irreducible sofic shift $\mathcal{X}$ is isomorphic to $S(\mathcal{X})$.

Example

Even shift

\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \draw[->] (1) edge[bend left] node[above] {$a$} (2);
  \draw[->] (2) edge[bend left] node[below] {$b$} (1);
  \draw[->] (1) edge node[above] {$b$} (2);
\end{tikzpicture}
\end{center}
The syntactic semigroup

The transition semigroup of the Krieger cover of a sofic shift $X$ is isomorphic to $S(X)$.

Example

Even shift

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\]

\[
\begin{array}{cc}
b & \text{3} \\
a & \text{1}
\end{array}
\]

\[
\begin{array}{cccc}
\ast[1,2,3] & \ast[2,1,3] \\
[2,1,3] & \ast[1,1,1]
\end{array}
\]

\[
\begin{array}{cc}
\ast[2,2] & \ast[2,2] \\
[2,2] & \ast[2,2]
\end{array}
\]

\[
\ast[\_,\_,\_]
\]
The Karoubi envelope

Let $S$ be a semigroup. The Karoubi envelope of $S$ is a category $K(S)$ defined by:

- the objects are the idempotents $e = e^2$ of $S$;
- the arrows $f \rightarrow e$ are the triples $(e, s, f)$ of elements of $S$ such that $s = esf$;
- the composition of arrows is given by

\[(e, s, f)(f, t, g) = (e, st, g)\].
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Note that it suffices to work with $LU(S) = \{s \in S \mid s = esf, e = e^2, f = f^2\}$:

$$
K(S) = K(LU(S)).
$$
Some remarks

- The Karoubi envelope of $S(\mathcal{X})$ is the Karoubi envelope of $S(\mathcal{X}) \setminus \{[2, 3, \_]\}$.

- The Karoubi envelope of an irreducible finite shift is equivalent to the Karoubi envelope of $\{0, 1\}$.

- The Karoubi envelope of the minimal shift is the trivial monoid.
Invariance of the Karoubi envelope

Denote by $K(X)$ the Karoubi envelope of $S(X)$.

**Theorem (Steinberg & AC)**

If $X$ and $Y$ are flow equivalent, then $K(X)$ and $K(Y)$ are equivalent.
A “monoidal” example

Corollary

If $\mathcal{X}$ and $\mathcal{Y}$ are flow equivalent shifts such that $S(\mathcal{X})$ and $S(\mathcal{Y})$ are monoids, then $S(\mathcal{X})$ and $S(\mathcal{Y})$ are isomorphic.

[Diagram of two flow equivalent shifts with labeled states and arrows indicating transitions between states.]
A "monoidal" example

A categorical invariant of flow equivalence
Extend the Krieger cover to a complete automaton by adding a sink state $\varnothing$.

Let $Q$ be the set of vertices of the extended automaton.

An arrow $(e, s, f)$ of $K(\mathcal{X})$ defines the mapping

$$A_{\mathcal{X}}(e, s, f) : Q \cdot e \rightarrow Q \cdot f$$

$$q \mapsto q \cdot s$$

The correspondence $(e, s, f) \mapsto A_{\mathcal{X}}(e, s, f)$ is a functor.
Invariance of the action

**Theorem**

*If $X$ and $Y$ are flow equivalent shifts, then the actions $A_X$ and $A_Y$ are equivalent.*

That is, there is an equivalence functor $F: K(X) \to K(Y)$ and a natural isomorphism $\eta: A_X \to A_Y \circ F$ such that...

\[
\begin{aligned}
Q(X)e & \xrightarrow{\eta_e} Q(Y)F(e) \\
A_X(e,s,f) & \downarrow \quad A_Y(F(e,s,f)) \\
Q(X)f & \xrightarrow{\eta_f} Q(Y)F(f).
\end{aligned}
\]
The proper communication graph

The proper communication graph of a (directed) graph $G$ is defined as follows:

1. take the set $PC(G)$ of nontrivial (i.e. having at least one edge) strongly connected components of $G$,
2. for $C_1, C_2 \in PC(G)$, make $C_1 \leq C_2$ if there is a path from a vertex of $C_1$ to a vertex of $C_2$,
3. the proper communication graph of $G$ is the acyclic graph induced by the poset $(PC(G), \leq)$.

**Theorem (Bates, Eilers & Pask, 2011)**

The proper communication graph of the Krieger cover of a sofic shift is a flow equivalence invariant.
The proper communication graph

Let \( LU(X) = \{ s \in S(X) \mid s = esf \text{, for some idempotents } e, f \in S(X) \} \).

Let \( q, r \) be vertices stabilized by some idempotent. Then \( q \leq r \) if and only if \( rLU(X) \subseteq qLU(X) \).

**Generalization**

The poset \( \{ qLU(X) \mid q \text{ is stabilized by some idempotent} \} \) is invariant under flow equivalence.
The action: an example
The action on the Fischer cover

**Theorem**

If $\mathcal{X}$ and $\mathcal{Y}$ are flow equivalent synchronizing shifts, then $\mathbb{K}(\mathcal{X})$ and $\mathbb{K}(\mathcal{Y})$ have equivalent actions over the corresponding Fischer covers.

![Fischer covers of two synchronizing shifts.](image)

**Figure:** Fischer covers of two synchronizing shifts.

In the first shift, the rank of every block of the shift is one or infinite. In the second shift, the word $ac$ acts as an idempotent of rank two.
An invariant hailing from Green’s relations

Legend for the label \((i, G, r)\) of a \(\mathcal{D}\)-class \(D\):

- \(i = 1\) if \(D\) is regular, \(i = 0\) otherwise;
- \(G\) is the (isomorphism class) of the Schützenberger group of \(D\);
- \(r\) is the rank of the action of the elements of the \(\mathcal{D}\)-class on the Fischer cover.

An analogous result holds replacing the Fischer cover by the Krieger cover, which is valid for all shifts (not necessarily sofic).
An invariant hailing from Green’s relations

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N. Jonoska constructed a conjugacy invariant for a reducible shift $X$, the poset of subsynchronizing subshifts of $X$.

These subshifts are the unions of subshifts whose finite blocks are factors of right contexts of a magic word for $L(X)$
(a word $m$ is magic if $m$ is synchronizing and $mA^* m \cap L(X) \neq \emptyset$).

Using our main result, we showed that this poset is a flow invariant.
Let $G$ be a finite (directed) graph. Let $G'$ be the graph obtained from $G$ by adjoining to each edge $x : u \to v$ an inverse edge $x^{-1} : v \to u$, establishing a bijection $x \in E(G) \mapsto x^{-1} \in E(G)^{-1}$.

**Presentation of the graph inverse semigroup $P_G$**

1. $P_G$ is generated by $E(G) \cup E(G)^{-1} \cup \{0\} \cup \{1_v \mid v \in V(G)\}$.
2. The generators are subject to the relations
   (a) $1_v$ are local identities
   (b) $xx^{-1} = 1_{\alpha x}$, $x \in E(G)$
   (c) $xy^{-1} = 0$ if $y \neq x$, $x, y \in E(G)$

- $P_G$ is an inverse semigroup.
- The words, over the alphabet $E(G) \cup E(G)^{-1}$, whose image in $P_G$ is not 0, define a shift, the **Markov-Dyck shift** $D_G$. 
Classification of Markov-Dyck shifts

If the out-degree of a vertex is always at least one, then the semigroup with zero $P_G$ is generated by $E(G) \cup E(G)^{-1}$.

**Lemma**

Suppose each vertex of $G$ has out-degree at least one. Then $P_G$ is the syntactic semigroup of $D_G$ if and only if $G$ has no vertex of in-degree exactly one.

**Theorem**

Let $G$ and $H$ with out-degree always at least one and in-degree never one.

$D_G$ is flow equivalent to $D_H \iff G \cong H$.

**Proof.**

- $D_G$ flow equivalent to $D_H \Rightarrow \mathcal{K}(P_G)$ equivalent to $\mathcal{K}(P_H)$
- $\mathcal{K}(P_G)$ equivalent to $\mathcal{K}(P_H) \Rightarrow G \cong H$