Problem Set # 2

WCATSS 2014

1. (a) Let A be a commutative Frobenius algebra over $\mathbb{C}$ of dimension n. Prove that there exist $e_1, \ldots, e_n \in A$ such that $e_i^2 = e_i$ for all i; $e_i e_j = 0$ if $i \neq j$; and $1 = e_1 + \cdots + e_n$.

   (b) Let $\tau \colon A \to \mathbb{C}$ be the trace and set $\mu_j = \tau(e_j)$. Suppose f is the 2-dimensional oriented field theory with $f(S^1) = A$. Compute $f(X_g) \in \mathbb{C}$, where X_g is the closed oriented connected surface of genus g.

2. This problem continues #2 from Problem Set #1, in particular uses the notation from that problem.

   (a) Suppose $X \colon Y_0 \to Y_1$ is a bordism. Construct a correspondence

$$\mathcal{C}(Y_0) \xleftarrow{\;s\;} \mathcal{C}(X) \xrightarrow{\;t\;} \mathcal{C}(Y_1)$$

   (b) Construct a field theory $F \colon \mathrm{Bord}_{\langle 1,2 \rangle} \to \mathrm{Vect}_{\mathbb{Q}}$ by "linearizing" the correspondences: to a closed 1-manifold Y let $F(Y)$ be the vector space of rational functions on $\mathcal{C}(Y)$, and then use "push-pull" to define F on a bordism. (That is, $F(X) = t_* \circ s^*$. You'll have to make sense of this!) Notice this works in any dimension, so in particular you recover the 1-dimensional theory from the first problem set.

   (c) Identify the Frobenius algebra $F(S^1)$. Complexify and apply problem #1. What are the e_i and μ_i in this case?

   (d) Recover a classical formula of Frobenius which counts the number of homomorphisms from the fundamental group of a surface of genus g to a finite group G.

3. Fix a field k and let Vect_k denote the symmetric monoidal category of vector spaces over k; the monoidal structure is given by tensor product. Prove that a vector space has a dual if and only if it is finite dimensional.

4. Are left and right duals (of objects) the same in braided monoidal categories?

5. Recall that for $B \to BO(n)$ a fibration, a B-structure on a smooth k-manifold ($k \leq n$) is a lift of its tangent classifier $M \xrightarrow{\tau_M} BO(k) \xrightarrow{-\times\mathbb{R}^{n-k}} BO(n)$ to B. (Warning: this definition is a little informal, since the tangent classifier τ_M is only defined up to contractible choices. There are a

number of ways to make this definition precise: choosing explicit models for everything in sight (for instance working with manifolds embedded into some Euclidean spaces, and working with Grassmanns), or using the language of stacks, or using ∞-categories).)

Fix a fibration $B \to BO(n)$ and a number $k < n$. Let P and Q be closed $(k-1)$-manifolds, each equipped with a B-structure. Let M be a k-dimensional B-cobordism from P to Q. Explain that there exists another k-dimensional B-cobordism $M^\vee$ from Q to P, and $(k+1)$-dimensional B-cobordisms E from $M \sqcup_Q M^\vee$ to $I \times P$ and C from $I \times Q$ to $M^\vee \sqcup_P M$. The full generality of B-structures isn't important here, so go ahead and do this for the two cases $B = BO(n)$ and $B = EO(n) \simeq *$ if you'd like; likewise, do this just for $(k,n) = (1,2)$ if you'd like.

6. The Kauffman bracket is an invariant of (unoriented) planar diagrams that takes values in $\mathbb{Z}[q, q^{-1}]$. It satisfies the relation

$$\langle RL \rangle - \langle Res_{TB} \rangle = q^{-1} \langle Res_{LR} \rangle$$

where here RL denotes the right-over-left crossing, Res_{TB} denotes the top-bottom resolution, and Res_{LR} denotes the left-right resolution.

   (a) Explain that the above relation, together with the specification $\langle U^{\sqcup k} \rangle = (q + q^{-1})^k$ for each $k \geq 0$, determine the values of $\langle - \rangle$ – here U denotes the unknot.

   (b) Verify that the Kauffman bracket is *not* a link invariant by showing that it is not invariant with respect to the Reidemister moves.

   (c) Verify that the correction $J(L) := (-1)^{\#LR} q^{2\#LR - \#RL} \langle D_L \rangle$ *is* an oriented link invariant.

   (d) Show that $J(L \sqcup U) = (q + q^{-1}) J(L)$.

   (e) Explain that J is the unique oriented link invariant for which $J(U^{\sqcup k}) = (q + q^{-1})^k$ and which satisfies the Skein relation:

$$q^2 J(LR) - q^{-2} J(RL) = (q - q^{-1}) J(Res)$$

   where LR is the left over right crossing, RL is the right over left crossing, and Res is the unique oriented resolution of the crossing.