

## Problem Set # 3

WCATSS 2014

1. Let  $G$  be a finite group,  $f$  the associated 2-dimensional finite gauge theory, and  $F$  the associated 3-dimensional finite gauge theory. These theories are defined on *unoriented* manifolds. (If we twist by a nonzero cohomology class, then orientations are required.)
  - (a) Compute  $f(M)$ , where  $M$  is the Möbius band. Note  $\partial M \simeq S^1$ , which we view as incoming, so  $f(M)$  is a linear functional on the vector space  $f(S^1)$ . Basis elements of  $f(S^1)$  correspond to irreducible complex representations of  $G$ , so we get a number for each such representation. Interpret the result in terms of the representation theory of  $G$ . Try particular examples, such as  $G = \mathbb{Z}/4\mathbb{Z}$ ,  $G = Q$ , where  $Q$  is the 8-element quaternion group.
  - (b) The finite path integral can be interpreted as an inverse limit (or just ‘limit’ in modern usage). Use this to compute  $f(pt)$ , which is a category. The groupoid of  $G$ -bundles on  $pt$  is  $*//G$ , the groupoid whose single object  $*$  has the group  $G$  of automorphisms. The finite path integral is the limit of the functor  $*//G \rightarrow \text{Cat}_{\mathbb{C}}$  which maps  $*$  to the category  $\text{Vect}_{\mathbb{C}}$  with trivial  $G$ -action. Here  $\text{Cat}_{\mathbb{C}}$  is the 2-category of linear categories. (Think informally about limits, if necessary.)
  - (c) Several variations: (i) include a nonzero cohomology class, which can be represented by a central extension  $\mathbb{T} \rightarrow \tilde{G} \rightarrow G$ ; (ii) replace  $\text{Cat}_{\mathbb{C}}$  by the 2-category of complex algebras, bimodules, and intertwiners; (iii) compute  $F(S^1)$  as a limit over the groupoid  $G//G$ , where  $G$  acts on  $G$  by conjugation; (iv) compute  $F(pt)$  as the limit of a functor into the 3-category of tensor categories.
2. In this problem you will use the cobordism hypothesis to compute tqfts in various cases.
  - (a) What are the 1-dimensional unoriented tqfts in the category of vector spaces? in super vector spaces? (Note the symmetry in the category of super vector spaces, which is an isomorphism  $V \otimes W \rightarrow W \otimes V$  for each pair of super vector spaces  $V, W$ , uses the Koszul sign rule.)
  - (b) Consider the super analog of the bicategory of algebras, bimodules, and maps. What are the fully-dualizable objects here? Is the Clifford algebra fully-dualizable? If so compute the Serre automorphism.
3. (a) Use the notation  $\tilde{C}(L)$  for the (graded) chain complex whose (bigraded) homology groups are the Khovanov homology groups of the oriented link  $L \subset \mathbb{R}^3$ . Use the notation  $L^\vee \subset \mathbb{R}^3$  for the link which is the composite embedding  $L \hookrightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3$  where the latter isomorphism is given by reflection about one of the coordinate planes. Show that  $C(L^\vee) \cong C(L)^\vee := \text{Hom}_{\mathbb{Z}}(C(L), \mathbb{Z})$  is the linear dual chain complex. Also, show that for disjoint links  $L, L'$  we have  $C(L \sqcup L') \cong C(L) \otimes C(L')$ .
  - (b) Write down a chain complex that computes the Khovanov homology of a trefoil (with a choice of orientation). Compute its homology.