Problem Set # 5

WCATSS 2014

1. Compute the Verlinde ring of $G = SU_2$ in terms of twisted equivariant K-theory as follows.
   (a) Show that there are two singleton conjugacy classes in G and every other conjugacy class is diffeomorphic to S^2. Deduce a G-equivariant cell decomposition with two 0-cells with stabilizer G and a single 1-cell with stabilizer $\mathbb{T}$, the diagonal subgroup of G. (A k-cell with stabilizer $H \subset G$ is homeomorphic to $B^k \times G/H$, where B^k is the k-dimensional open ball.)
   (b) Recall that the equivariant K-theory group $K_H(\mathrm{pt})$ is isomorphic to the complex representation ring $R(H)$. Identify $R(SU_2)$, $R(\mathbb{T})$, and the restriction map $R(SU_2) \to R(\mathbb{T})$ in terms of polynomial and Laurent rings.
   (c) In general an equivariant cell structure gives a spectral sequence for generalized cohomology, but because we have only 0- and 1-cells the spectral sequence collapses to a short exact sequence. Compute the untwisted equivariant K-theory group $K_G(G)$.
   (d) A twisting is an equivariant line bundle over the 1-cell, so a character of $\mathbb{T}$. Let the twisting τ be the character $\lambda \mapsto \lambda^r$, $\lambda \in \mathbb{T}$, for some positive integer r. Compute $K_G^\tau(G)$. Express the result as a quotient of $R(SU_2)$.

2. Let $S^1 \xrightarrow{e} \mathbb{R}^3 \xleftarrow{e'} S^1$ be two disjoint knots. Let A be an E_3-algebra in $(\mathrm{Ch}_\mathbb{Q}, \otimes)$, the symmetric monoidal category of chain complexes over the rationals with tensor product. Use the defining properties of factorization homology to construct the sequence of maps

$$C_*(S^1 \times S^1; \mathbb{Q}) \otimes (A \otimes A) \to \int_{S^1 \sqcup S^1} A \to \int_{\mathbb{R}^3} A \simeq A\ .$$

   Show that this composite map factors as the map $(A \otimes A)[2] \xrightarrow{\cdot \mathrm{Lk}(e,e')} (A \otimes A)[2] \xrightarrow{[-,-]} A$, which is the linking number of e and e' times the Lie bracket operation on A.

3. Let M be a closed connected oriented 3-manifold equipped with a self-indexing Morse function $f \colon M \to \mathbb{R}$. As in problem 6 on Problem Set #4, such an f determines Heegaard data: $(\Sigma_g; \alpha, \beta)$.
   (a) Explain that a point in the intersection $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^g(\Sigma_g)$ is the same data as a choice of gradient flow lines connecting all intermediate critical points of M.
   (b) Removing a neighborhood of these flow lines, as well as a neighborhood of a flow line between the extremal critical points of f, results in a 3-manifold with boundary which is a disjoint union of 2-spheres, equipped with a non-vanishing vector field on it. Explain why the vector field on each boundary component has index zero, and so can be extended to a non-vanishing vector field on the entire 3-manifold.

(c) Convince yourself that a holomorphic strip in $\mathrm{Sym}^g(\Sigma)$ connecting two such intersection points exists only if these two (not-uniquely) associated non-vanishing vector fields are connected by a path among such, at least after removing some open balls from the 3-manifold.

(d) Conclude that (each variant of the) Heegaard-Floer chain complex split as a direct sum indexed by equivalence classes of non-vanishing vector fields on the 3-manifold, where two are deemed equivalent if they are homotopic on the 3-manifold without some balls.

(e) Consider the Lie group $\mathrm{Spin}^c(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}/2} U(1)$ with the canonical central actions of $\mathbb{Z}/2$. The *moduli space of* Spin^c*-structures* on a rank n vector bundle $E \to B$ is the space of lifts of its classifying map $B \to EBO(n)$ to $B\mathrm{Spin}^c(n)$, other moduli spaces are similar. Show that the moduli space of Spin^c-structures on $E \to B$ is the moduli space of pairs $(L \to B, B \xrightarrow{E \oplus L} B\mathrm{Spin}(n+2))$ of complex line bundles on B together with Spin-structures on the direct sum. Show also that the moduli space of Spin^c-structures on an oriented rank n vector bundle $E \to B$ is the space of lifts of the composite map $w_2 \colon B \to EBSO(n) \to B^2\mathbb{Z}/2$, the latter which classifies the universal cover of $SO(n)$ $(n > 2)$, to $B^2\mathbb{Z}$ along the "modulo 2" map.

(f) Now take $n = 3$. Show that $BSO(2) \to BSO(3) \to B^2\mathbb{Z}/2$ canonically factors through $B^2\mathbb{Z}$. Identify the fiber of the map $BSO(3) \to B^2\mathbb{Z}/2$ as $B^2\mathbb{Z}$ using the Hopf fibration. Use the previous two sentences to argue that there is a bijection between the set of path components of the moduli space of Spin^c-structures on a closed oriented 3-manifold, and the set of equivalences classes of non-vanishing vector fields on the 3-manifold, where two such are equivalent if they are homotopic on the 3-manifold minus some open balls.