“Confluence” in Ito-Sadahiro number systems

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For $\beta > 1$ we want to write numbers in the form $\sum_{i \leq N} a_i \beta^i$.

Let

$$T_\beta : [0, 1) \to [0, 1), \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor$$

Then

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \ldots, \quad \text{where } x_i = \lfloor \beta T^{i-1}(x) \rfloor$$

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Admissibility condition

Theorem (W. Parry)

A sequence \( x = x_1x_2x_3 \ldots \) is \( \beta \)-admissible iff for each \( i \geq 1 \)

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0^\omega \preceq_{\text{lex}} x_1x_{i+1}x_{i+2} \cdots \preceq_{\text{lex}} \lim_{\varepsilon \to 0^+} d_\beta(1 - \varepsilon).
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Ordering on \( \mathbb{R} \) corresponds to the lexicographic ordering of \( d_\beta(x) \).

Expansion \( d_\beta(x) \) is the biggest amongst all the representations in lexicographic order.
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Ordering on $\mathbb{R}$ corresponds to the lexicographic ordering of $d_\beta(x)$.

Expansion $d_\beta(x)$ is the biggest amongst all the representations in lexicographic order.
Now for $\beta > 1$ we would like to write numbers as $\sum_{i\leq N} a_i(-\beta)^i$.

Let $\mathcal{I} = \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right) = [\ell, \ell + 1)$ and

$$T_{-\beta} : \mathcal{I} \rightarrow \mathcal{I}, \quad T_{-\beta}(x) = -\beta x - \lfloor -\beta x - \ell \rfloor$$

Then

$$x = \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \frac{x_3}{(-\beta)^3} + \cdots, \quad \text{where } x_i = \lfloor -\beta T^{i-1}(x) - \ell \rfloor.$$

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Theorem (S. Ito, T. Sadahiro)

A string $x_1x_2x_3 \ldots$ is $(-\beta)$-admissible iff for each $n \geq 1$

$$d_{-\beta}(\ell) \preceq_{\text{alt}} x_i x_{i+1} x_{i+2} \cdots \prec_{\text{alt}} \lim_{\varepsilon \to 0^+} d_{-\beta}(\ell + 1 - \varepsilon)$$

When $x \notin I$, we divide by a suitable power of $(-\beta)$ and expand $x/(-\beta)^k$.

When $d_{-\beta}(x/(-\beta)^k) = x_1x_2 \ldots$, we denote

$$\langle x \rangle_{-\beta} = x_1 \ldots x_k \bullet x_{k+1} \cdots \approx x_1(-\beta)^{k-1} + \cdots + x_k(-\beta)^0 + \ldots$$
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We define $(\pm \beta)$-integers as

$$Z_\beta = \{ x \in \mathbb{R} \mid \langle |x| \rangle = x_1 \ldots x_k \bullet 0^\omega \} = \bigcup_{n \geq 0} \beta^n T_\beta^{-n}(0)$$

$$Z_{-\beta} = \{ x \in \mathbb{R} \mid \langle x \rangle = x_1 \ldots x_k \bullet 0^\omega \} = \bigcup_{n \geq 0} (-\beta)^n T_{-\beta}^{-n}(0)$$

By coding gaps in $Z_{-\beta}$ by letters of an alphabet, one gets a bidirectional infinite word $u_\beta$, resp. $u_{-\beta}$.

The words $u_\beta$ and $u_{-\beta}$ are invariant under substitution.

Substitutions are over a finite alphabet for $d_\beta(1)$, resp $d_{-\beta}(\ell)$

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Some properties of $(\pm \beta)$-integers

Unlike $\mathbb{Z}_\beta$ is $\mathbb{Z}_{-\beta}$ not symmetric around 0.

$\mathbb{Z}_{-\beta} = \{0\}$ iff $\beta < \frac{1+\sqrt{5}}{2}$. This never happens for $\mathbb{Z}_\beta$.

$\mathbb{Z}_\beta$ is relatively dense, i.e. lengths of gaps are $< K$.

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Motivation

Lemma

Let $\beta > 1$ be root of $x^2 - mx - m$, $m \geq 1$. Then

$$\mathbb{Z}_{-\beta} = \left\{ \sum_{i \geq 0} a_i(-\beta)^i \mid a_i \in A_{-\beta} \right\}$$

For $\beta$-numeration, we have the following theorem

Theorem (Ch. Frougny)

Let $\beta > 1$ then the following conditions are equivalent:

1. $\beta$ is root of $x^k - mx^{k-1} - \cdots - mx - n$ for $m \geq n \geq 1$.
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Let $\beta > 1$. Then three following conditions are equivalent:

1. $\beta$ is root of $x^k - mx^{k-1} - \cdots - mx - n$, where $m \geq n \geq 1$ and $m = n$ for $k$ even.

2. $\mathbb{Z}_{-\beta} = \left\{ \sum_{i \geq 0} a_i(-\beta)^i \mid a_i \in A_{-\beta} \right\}$.

3. Substitutions fixing $u^+_{\beta}$ and $u^-_{\beta}$ are conjugate.
You will see $1) \Rightarrow 2)$ and consequently $1) \Rightarrow 3)$ on the blackboard.
Confluence property implies spaces in $\mathbb{Z}_{-\beta}$ are $\leq 1$.

It follows that $d_{-\beta}(\ell) = m0 m0 \ldots m0 a b \ldots$, $ab \neq m0$.

We take the shortest forbidden string $1m0m0 \ldots 0m$ where $m$ is the maximal digit.
Proof continued: 2) $\Rightarrow$ 1)

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One can show that admissible transcription is of the form

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= & 0 & 0 & a_1 & a_2 & a_3 & \ldots & a_k & \bullet
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In $\beta$-systems, rewriting system associated to $\beta$ was confluent.

The $(-\beta)$-rewriting system is not confluent, e.g. for $\beta = \frac{1 + \sqrt{5}}{2}$ we have

\[ 1\bullet = 110\bullet = 11010\bullet = \ldots \]

Arithmetics of confluent $\pm \beta$?

- If $\beta$ is $+$confluent then the set of numbers with finite expansion forms a ring.
- If $\beta$ is $-$confluent then $m + 1$ has infinite expansion.
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“Confluent” bases appear in:

- Study of optimal representations (K. Dajani et al.)
- Study of Rauzy fractals and reversal invariant language of $u_\beta$ (J. Bernat)
- Description of spectra of numbers (D. Garth & K. Hare)

Study of the set

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Comparison of $\beta$- and $(-\beta)$- numeration

Ch. Kalle: Let $\beta \in (1, 2)$ then $T_\beta$ and $T_{-\beta}$ are measurably isomorphic iff $\beta$ is root of $x^k - x^{k-1} - \cdots - x - 1$.

Conjecture: This holds also for roots of

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