1. A 2-Dimensional Reduction of Chern-Simons

Quantum Chern-Simons theory is a (3,2,1)-dimensional TFT on oriented manifolds with a $p_1$-structure with values in the 2-category of $\mathbb{C}$-linear categories,

$$Z_C : \text{Bord}_{(3,2,1)}^{(w_1,p_1)} \to \text{Cat}_\mathbb{C},$$

with

$$Z_C(S^1) \simeq \mathcal{C},$$

a modular tensor category. That is, a ribbon fusion category with a non-degenerate S-matrix. In particular, $\mathcal{C}$ is linear (over $\mathbb{C}$), braided, has duals, and is semisimple with finitely many simple objects.

**Remark 1.1.** A $p_1$-structure on a manifold $M$, is the data of a null homotopy of the composition

$$M \to BO \to K(\mathbb{Z}, 4),$$

where $M \to BO$ classifies the (stable) tangent bundle of $M$, and $BO \to K(\mathbb{Z}, 4)$ is the first Pontryagin class.

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Remark 1.2. Philosophically, a modular tensor category is a categorification of a commutative Frobenius algebra. If \( \mathcal{C} \) is a modular tensor category, then \( K^0(\mathcal{C}) \) inherits the structure of a commutative ring over \( \mathbb{Z} \) from the braiding on \( \mathcal{C} \), and thus, \( K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} \) is a commutative algebra. The trace map \( K^0(\mathcal{C}) \to \mathbb{C} \) sends the equivalence class \( V \) to \( \dim(V) := \text{Trace}(\text{Id}_V) \), the latter of which is defined in any ribbon tensor category.

Definition 1.3. The Verlinde ring is \( K^0(\mathcal{C}) \), and the Verlinde Algebra of \( \mathcal{C} \) is the algebra \( K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} \). The former is a Frobenius ring while the latter is a Frobenius algebra.

Two main examples of interest are the following:

Example 1.4. Let \( G \) be a finite group, then \( \text{Vect}_G(G) \), the category of \( G \)-equivariant vector bundles on \( G \), is a modular tensor category. The monoidal structure is defined as follows. Let \( V \) be an equivariant vector bundle on \( G \), that is a collection of vector spaces \( V_x \), \( x \in G \), and isomorphisms \( V_x \cong V_{gxg^{-1}} \) satisfying a cocycle condition. Given two equivariant vector bundles \( V \) and \( W \), we define a new equivariant vector bundle \( V \otimes^c W \) using convolution:

\[
(V \otimes^c W)_x := \bigoplus_{x_1x_2=x} V_{x_1} \otimes W_{x_2}.
\]

Notice that,

\[
(V \otimes^c W)_{gxg^{-1}} := \bigoplus_{x_1x_2=gxg^{-1}} V_{x_1} \otimes W_{x_2} \\
\cong \bigoplus_{g^{-1}x_1x_2g=x} V_{x_1} \otimes W_{x_2} \\
\cong \bigoplus_{g^{-1}x_1gg^{-1}x_2g=x} V_{g^{-1}x_1} \otimes W_{g^{-1}x_2} \\
\cong \bigoplus_{y_1y_2=x} V_{y_1} \otimes W_{y_2} \\
\cong (V \otimes^c W)_x
\]
So that $V \otimes c W$ is indeed another equivariant vector bundle. One can show, $K^0(\mathcal{C}) = K_G(G)$ is a Frobenius algebra, multiplication arises from the pushforward of group multiplication.

**Example 1.5.** There is a twisted version of the example above. Let 
\[ \alpha \in H^4(BG, \mathbb{Z}) \to H_3^G(G, \mathbb{Z}) \simeq H^2_G(G, U(1)) \simeq H^1_G(G, \{\text{Line Bundles}\}) \]
where the first arrow in the sequence sends a map $BG \to B^4\mathbb{Z}$ to a map $G/G \to B^2\mathbb{C}^\times \times B^3\mathbb{C}^\times \to B^2\mathbb{C}^\times$, by taking free loops. That is, we get a $\mathbb{C}^\times$-gerbe on $G/G$. For the remaining arrows, recall that
\[ \mathbb{Z} \simeq K(\mathbb{Z}, 0), \ U(1) \simeq K(\mathbb{Z}, 1), \ \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2). \]

From the data of $\alpha$, one can construct hermitian lines $L_{x,y}$ with isomorphisms $L_{yxy^{-1},z} \otimes L_{x,y} \to L_{x,zy}$, where $x, y, z \in G$. Then $\text{Vect}_G^\alpha(G)$, the category of $\alpha$-twisted equivariant vector bundles on $G$, is a modular tensor category with a monoidal structure given by $\alpha$-twisted convolution. An object in this category is a vector bundle $V$ over $G$ together with isomorphisms $L_{x,y} \otimes V_x \to V_{yxy^{-1}}$, where the $L_{x,y}$ are hermitian lines constructed using the data of $\alpha$. These are equivariant vector bundles twisted by a gerbe.

**Definition 1.6.** Now, let $G$ be simply connected, compact, simple Lie group and let 
\[ 1 \to \mathbb{C}^\times \to \tilde{LG} \to LG \to 1 \]
be the universal central extension corresponding to a generator of $H^2(LG, \mathbb{C}^\times)$.

**Remark 1.7.** A projective representation of $LG$ is equivalent to a honest representation of $\tilde{LG}$, where we require the center $\mathbb{C}^\times$ to act by scalar multiplication.

**Definition 1.8.** A positive energy representation of $LG$ at level $\alpha$ is a representation of $\tilde{LG}, V$, extending to the semi-direct product $\tilde{LG} \rtimes \text{Rot}(S^1)$, such that $\text{Rot}(S^1)$ acts by non-negative characters only. That is, $\text{Rot}(S^1)$ induces a decomposition of vector spaces 
\[ V = \bigoplus_{n \geq 0} V(n) \]
where \( V(n) = \{ v \in V | R_\theta v = e^{in\theta} v \} \) and \( R_\theta \in \text{Rot}(S^1) \).

**Remark 1.9.** If \( V \) is irreducible, the kernel of the central extension acts by a single scalar \( \alpha \) (Schur’s Lemma), called the level of the representation. The level classifies the central extension of \( LG \) and is a class \( \alpha \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z} \). Furthermore, \( V \) is determined by its level and its lowest nonzero energy eigenspace, which itself is an irreducible representation of \( G \). We will use this fact in the sequel.

**Proposition 1.10.** Given \( G \) and an element \( \alpha \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z} \), the category of positive energy representations of the loop group \( LG \) at level \( \alpha \), \( \text{Rep}^\alpha(LG) \), is a modular tensor category.

**Definition 1.11.** Let \( \text{Ver}_\alpha(G) \) be the Verlinde ring of the modular tensor category \( \text{Rep}^\alpha(LG) \), and let \( \text{Ver}_\alpha(G) \otimes \mathbb{Z} \mathbb{C} \) be its Verlinde algebra.

In this talk, we will consider a 2-dimensional reduction of Chern-Simons theory. This is an oriented 2-dimensional TFT \( Z'_C \) defined by

\[
Z'_C(M) := Z_C(S^1 \times M).
\]

In particular,

\[
Z'_C(\text{pt}) := Z_C(S^1 \times \text{pt}) \simeq \mathcal{C}.
\]

\[
Z'_C(S^1) \simeq \text{HH}_0(\mathcal{C}).
\]

**Remark 1.12.** We consider an oriented 2-dimensional TFT because the map defining the 2-dimensional reduction

\[
\text{Bord}^{(w_1, p_1)}_2 S^1 \times - \longrightarrow \text{Bord}^{(w_1, p_1)}_{(3,2,1)} \longrightarrow \text{Cat}_\mathbb{C}
\]

factors through the oriented bordism category \( \text{Bord}^{w_1}_2 \).

We claim there is a commutative diagram

\[
\begin{array}{ccc}
\text{Bord}^{(w_1, p_1)}_{(3,2,1)} & \longrightarrow & \text{Cat}_\mathbb{C} \\
\text{Bord}^{(w_1, p_1)}_2 & \longrightarrow & \text{Bord}^{w_1}_2 \\
\text{S}^1 \times & \text{s.l.} & \\
\end{array}
\]

**Goal 1.13.** Show the Verlinde Algebra \( K^0(\mathcal{C}) \otimes \mathbb{Z} \mathbb{C} \) is the Frobenius algebra defining the \((2,1)\)-dimensional reduction \( Z'_C \).
Thus, we must show the following

**Proposition 1.14.** Let $\mathcal{C}$ be a modular tensor category, then

$$K^0(\mathcal{C}) \otimes \mathbb{Z} \mathcal{C} \simeq \text{HH}_0(\mathcal{C}).$$

**Proof.** There is an isomorphism $K^0(\mathcal{C}) \otimes \mathbb{Z} \mathcal{C}$ with the algebra of $\mathbb{C}$-valued functions on the finite set $I$ of isomorphism classes of simple objects. This uses the non-degeneracy of the S-matrix [Bakalov-Kirillov 3.1.12]. This algebra can be interpreted as $\text{End}(\text{Id}_\mathcal{C})$ by Schur’s Lemma. Furthermore, $\text{End}(\text{Id}_\mathcal{C}) \simeq \text{HH}_0(\mathcal{C})$, using the semisimplicity of the category $\mathcal{C}$. \qed

2. **The example $G = SU(2)$ and $\alpha = k$**

Start with the complexified representation ring $\text{Rep}(SU(2)) = \mathbb{C}[t, t^{-1}]^{\Sigma_2}$. That is, the irreducible representations are $V_n$ with $\dim(V_n) = n + 1$. This representation corresponds to the polynomial $t^n + t^{n-2} + \ldots + t^{-n}$. Multiplication of polynomials gives the formula:

$$V_n \otimes V_m = V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{|m-n|}.$$

The Verlinda algebra, $\text{Ver}_k(SU(2)) \otimes \mathbb{Z} \mathbb{C}$, is a quotient of $\text{Rep}(SU(2))$ by

$$V_{k+1} = 0$$

and the relation

$$V_n \oplus V_{2k+2-n} = 0.$$

**Example 2.1.** Take $k = 5$, then in $\text{Rep}(SU(2))$. Draw a picture with a mirror at 6!

$$V_3 \otimes V_4 = V_7 \oplus V_5 \oplus V_3 \oplus V_1$$

and in the quotient $\text{Ver}_5(SU(2))$ this becomes

$$V_3 \otimes V_4 = -V_5 \oplus V_5 \oplus V_3 \oplus V_1 = V_3 \oplus V_1.$$

If $k = 0$ we have

$$\text{Ver}_0(SU(2)) \otimes \mathbb{Z} \mathbb{C} = \mathbb{C}[x]/x.$$

If $k = 1$ we have

$$\text{Ver}_1(SU(2)) \otimes \mathbb{Z} \mathbb{C} = \mathbb{C}[x]/(x^2 - 1).$$
One can show:

\[ Ver_{k-1}(SU(2)) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[x]/(\prod_{m=1}^{k} (x - 2 \cos(\frac{m}{2k + 2\pi}))). \]

**Remark 2.2.** Again, positive energy representations of level \( k \) are determined by their lowest energy eigenstate which itself is an irreducible representation of the group, in this case \( SU(2) \). The first equation \( V_{k+1} = 0 \) corresponds to the fact that the antidominant weights controlling irreducible representations live in the positive Weyl alcove [Segal-Pressley 9.3.5].

**Remark 2.3.** The unit of the algebra is called the Vacuum representation of level \( k \). It is the positive energy representation of level \( k \) that is induced from a lowest energy eigenspace being a lowest weight representation.

**Remark 2.4.** The “fusion” algebra structure has origins in conformal field theory. Let \( V_p, V_q \) be irreducible positive energy representations of \( G = SU(2) \). Then,

\[ V_p \cdot V_q = \sum_{V_r} N_{V_p, V_q}^{V_r} V_r, \]

Here, \( N_{V_p, V_q}^{V_r} \) is the dimension of the vector space

\[ (V_p \otimes V_q \otimes V_r^*)_{\text{Hol}(\mathbb{P}^1 - \{p_1, p_2, p_3\}, G_{\mathbb{C}})}. \]

This is (dual to) the space of conformal blocks. One can show this multiplication is associative and gives rise to the Verlinde algebra. There is a subtle point here. Let \( \hat{G} \) be the canonical central extension of \( \text{Hol}(\mathbb{C}^*, G_{\mathbb{C}})^{\times 3} \) that extends each of the individual universal central extensions. Then, one needs to show the image of

\[ \text{Hol}(\mathbb{P}^1 - \{p_1, p_2, p_3\}, G_{\mathbb{C}}) \to \hat{G} \]

splits to actually have a well defined action of \( \text{Hol}(\mathbb{P}^1 - \{p_1, p_2, p_3\}, G_{\mathbb{C}}) \) on \( V_p \otimes V_q \otimes V_r^* \). This is established by using the residue formula.

Furthermore, one can show that:

\[ N_{V_p, V_q}^{V_r} = \begin{cases} 1 & \text{if } r - |p - q| \text{ is even and } |p - q| \leq r \leq \min(p + q, 2k - p - q) \\ 0 & \text{otherwise} \end{cases} \]
This computation is in Verlinde’s original paper.

3. Twistings and Orientations

To give a complex vector bundle on $M$ is to give vector bundles $V_i$ on open sets $U_i$ of a covering and isomorphisms

$$\lambda_{ij} : V_i \to V_j$$

which satisfy a cocycle condition on intersections. In complex $K$-theory this is expressed by the Mayer-Vietoris sequence. In forming a twisted vector bundle $V$, one introduces a complex line bundle $L_{ij}$ on $U_i \cap U_j$ together with isomorphisms:

$$\lambda_{ij} : L_{ij} \otimes V_i \to V_j.$$  

The $L_{ij}$ must come equipped with isomorphisms

$$L_{jk} \otimes L_{ij} \to L_{ik}$$

on triple intersections and satisfy a cocycle condition on quadruple intersections. Thus, we can form a twisted version of $K(M)$ given an element $\tau \in H^1(M, \{\text{Line Bundles}\}) \cong H^3(M, \mathbb{Z})$. This group parametrizes complex $\mathbb{C}^\times$-gerbes.

**Remark 3.1.** There is second way to think about this. Recall that $\mathcal{F}$ (the space of Fredholm operators of a complex Hilbert space $\mathcal{H}$) is a representing space for $K$-theory. That is, $K(X) = \pi_0 \Gamma(X \times \mathcal{F} \to X)$. If $U = U(\mathcal{H})$ is the unitary group and $P \to X$ is a principal $PU$-bundle, one can form the associated bundle $\xi = P \times_{PU} \mathcal{F} \to X$ with fiber $\mathcal{F}$. Define $P$-twisted $K$-theory to be

$$K(X)_P = \pi_0 \Gamma(\xi \to X).$$

Thus one twists $K$-theory by $PU$-bundles over $X$, and isomorphism classes of such bundles are given by $[X, BPU]$. Since $BPU$ is a model for $K(\mathbb{Z}, 3)$, we arrive at $[X, BPU] \cong H^3(X, \mathbb{Z})$.

**Remark 3.2.** A third way to think about this from an $\infty$-categorical perspective is in Ando-Blumberg-Gepner’s “Twists of $K$-Theory.” They
discuss a map \( K(\mathbb{Z}, 3) \xrightarrow{T} BGL_1(K) \cong |\text{Line}_K| \) and form the composition

\[
M \xrightarrow{T} K(\mathbb{Z}, 3) \xrightarrow{T} BGL_1(K) \cong |\text{Line}_K|.
\]
The corresponding Thom spectrum is

\[
M^{T\tau} := \text{colim}(\text{Sing}M \xrightarrow{T\tau} \text{Line}_K \xrightarrow{} \text{Mod}_K).
\]
Finally, twisted \( K \)-theory is given by

\[
K^n(M^\tau) := \pi_0(\text{Mod}_K(M^{T\tau}, \Sigma^n K)).
\]

Remark 3.3. That is, twistings of \( K \)-theory on a space \( M \) are classified up to isomorphism by the set \( H^0(M, \mathbb{Z}/2) \times H^1(M, \mathbb{Z}/2) \times H^3(M, \mathbb{Z}) \).

Example 3.4. A real vector bundle \( V \rightarrow M \) determines a twisting \( \tau_V \) in complex \( K \)-theory, whose equivalence class is:

\[
[\tau_V] = (\text{rank} V, w_1(V), W_3(V)) \in H^0(M, \mathbb{Z}/2) \times H^1(M, \mathbb{Z}/2) \times H^3(M, \mathbb{Z})\)
\]

More precisely, a real vector bundle \( V \) has a second Stiefel-Whitney class \( w_2(V) \in H^2(M, \mathbb{Z}/2) \), and gives a real \( \mathbb{R}^\times \)-gerbe. The third integral Stiefel-Whitney class \( W_3(V) \) is the image of \( w_2(V) \) under the Bockstein \( H^2(M, \mathbb{Z}/2) \rightarrow H^3(M, \mathbb{Z}) \), and corresponds to complexification.

One can further assign a twisting to any virtual real vector bundle by setting \( \tau_{-V} := -\tau_V \).

Remark 3.5. Let \( \tau \) be a twisting on a manifold \( N \), then to a proper map \( p : M \rightarrow N \) one can define a pushforward map

\[
p_* : K^{(\tau_p + p^*\tau)}(M) \rightarrow K^{\tau + *}(N).
\]
where \( \tau_p = \tau_M - p^*\tau_N \) is the twisting associated to the relative tangent bundle.
Definition 3.6. A $KU$-orientation of $V$ is an equivalence $\tau_V \simeq \tau_{\text{rank}V}$ or equivalently, a trivialization of the twisting attached to the reduced bundle $(V - \text{rank}V)$. Equivalently, this is a $\text{spin}^c$ structure on $V$.

Definition 3.7. An orientation of a manifold or stack is an orientation of its (virtual) tangent bundle. Recall that the tangent bundle to a smooth stack is a graded vector bundle. We form a virtual bundle by taking the alternating sum of its homogeneous components.

Definition 3.8. A $KU$-orientation of a map $p : M \to N$ is a trivialization of the twisting $\tau_{TM - p^*TN - \text{rank}_p} = \tau_p - \tau_{\text{rank}_p}$. Thus, to a $KU$-oriented, proper map $p : M \to N$ one can define a pushforward map

$$p_* : K^\bullet + \text{dim}(M) - \text{dim}(N)(M) \to K^\bullet(N).$$

Example 3.9. If $X$ is a closed oriented 2-manifold, the tangent space of the stack $M_X$ at $A$ (a connection on principal bundle $P$) is the complex

$$0 \to \Omega_X^0(g_P) \xrightarrow{d_A} \Omega_X^1(g_P) \xrightarrow{d_A} \Omega_X^2(g_P),$$

where $g_P$ is the adjoint bundle associated to $P$. One forms the virtual tangent bundle to $M_X$ as the index of an elliptic complex. The reduced tangent bundle to $M_X$ is computed by the de Rham complex coupled to the reduced adjoint bundle $\overline{g}_P := g_P - \text{dim}G$.

Freed-Hopkins-Teleman describe a universal orientation that simultaneously orients $M_X$ for not only closed 2-manifolds $X$, but 2-manifolds with boundary. In particular, the restriction maps $t : M_X \to M_{\partial X}$ are oriented. That is, there is a trivialization of the twisting $\tau_t - \tau_{\text{rank}t}$.

4. Pushforward Using Consistent Orientations

Let $G$ be a compact Lie group. Let $Z$ be a 1 or 2 dimensional oriented manifold. Let $M_Z$ be the stack of flat connections of $X$. For example, $M_{S^1} \simeq G/G$. To a bordism $X : Y_0 \to Y_1$ we consider the correspondence of flat $G$-connections:
We would like to define a push-pull

\[ Z_X := t_* \circ s^* : K^\bullet(M_{Y_0}) \to K^\bullet(M_{Y_1}). \]

But the pushforward, \( t_* \), requires an orientation on (twisted) K-theory. Freed-Hopkins-Teleman show that orientations can be consistently chosen. Moreover, the functor \( Z \) respects gluing, i.e. is functorial and defines a 2d-TFT. For instance, given

We have that

\[ (t'r')_* \circ (sr)^* = [t'_* \circ s'^*] \circ [t_* \circ s^*] \]

**Remark 4.1.** Moreover, they show there is a well-defined map from “consistent orientations” to levels on \( G \).

**Remark 4.2.** Notice that \( M_{S^1} \simeq G/G \) as stacks, and thus \( K^\bullet(M_{S^1}) \simeq K^\bullet_G(G) \).

**Theorem 4.3.** For any compact Lie group \( G \), once a consistent orientation is chosen (and hence a level), the value of \( S^1 \) on the corresponding 2d TFT recovers the Frobenius ring

\[ K^{ullet \theta + h + \alpha}_G(G) \simeq Ver_\alpha(G). \]
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