Strong shift equivalence of matrices over a ring

joint work in progress with Scott Schmieding

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Introduction

For problems of $\mathbb{Z}^d$ SFTs and their relatives:

$d \geq 2$: 
computability conditions are fundamental.

$d = 1$: 
Key features:  
(1) Algebra around matrices  
(SSE, SE, related invariants)  
(2) Positivity constraints

This talk reports progress on (1).

All rings and semirings are assumed to contain \{0, 1\}. 
Strong shift equivalence

Let $S$ be a semiring.
And on the first day [1973], Williams defined strong shift equivalence.

Matrices $A, B$ over $S$ are elementary strong shift equivalent over $S$ (ESSE-$S$) if they are square and there exist matrices $U, V$ over $S$ such that

$$A = UV \quad \text{and} \quad B = VU.$$ 

$A, B$ are strong shift equivalent over $S$ (SSE-$S$) if there exists a chain

$$A = A_0, A_1, \ldots, A_\ell = B$$

with $A_{i-1}$ and $A_i$ ESSE-$S$ for $0 < i \leq \ell$. 
Why did Williams define SSE?

- Up to topological conjugacy, every shift of finite type (SFT) is an “edge SFT” $\sigma_A$, defined by a square matrix $A$ over $\mathbb{Z}_+$. 

- $\sigma_A$ and $\sigma_B$ are isomorphic (topologically conjugate) iff $A, B$ are SSE-$\mathbb{Z}_+$. 

But SSE over $\mathbb{Z}_+$ is very hard to understand completely (not known to be decidable, even restricted to small cases).

So on the second day, Williams defined ...
Shift equivalence

DEFN Square matrices $A, B$ are shift equivalent over $S$ (SE-$S$) if $\exists$ matrices $U, V$ over $S$ and $\ell \in \mathbb{N}$ such that

$$A^\ell = UV \quad B^\ell = VU$$

$$AU = UB \quad BV = VA$$

Always: SSE-$S$ implies SE-$S$. Also

- SE-$\mathbb{Z}_+$ is decidable (Kim-Roush).

- SE-$\mathbb{Z}_+$ turns out to be reasonably tractable, and closely related to significant applications in symbolic dynamics

- SE over $\mathbb{Z}$ (or other rings) has useful and conceptually satisfying algebraic reformulations.
Classifying shifts of finite type.

Williams gave us:

- **Theorem (Annals of Math 1973)**
  \[ SE-\mathbb{Z}_+ \implies SSE-\mathbb{Z}_+ . \]

- **Conjecture (Annals of Math 1974)**
  \[ SE-\mathbb{Z}_+ \implies SSE-\mathbb{Z}_+ . \]

Eventually counterexamples were constructed (Kim Roush 1992,1999), based on a lovely algebraic topological structure created by Wagoner ("strong shift equivalence space"). No progress since on understanding refinement of \( SSE-\mathbb{Z}_+ \) by \( SE-\mathbb{Z}_+ \).

However ...

From here \( S \) is a ring.
There are good reasons to study SSE over other rings and semirings.

- To approach the $\mathbb{Z}$ problem.

- There are other symbolic dynamical systems presented by matrices over $S_+$ and classified up to conjugacy by SSE over $S_+$. E.g.:

  $\mathcal{S} = \mathbb{Z}G$, $G$ finite: SSE-$\mathbb{Z}_+G$ classifies free $G$-SFTs.

  $\mathcal{S} = \mathbb{Z}G$, $G = \mathbb{Z}^n$: SSE-$\mathbb{Z}_+G$ classifies irred. SFTs with Markov measure.

  $\mathcal{S} =$ integral semigroup ring of a certain noncommutative semigroup: SSE over $S_+$ classifies sofic shifts.
• For understanding constraints of order on algebraic properties of matrices.

• Understanding SSE-$S$ for its own sake.

• Understand better proofs that can’t work and theorems that can’t be proved.

Before confronting the hard problem of understanding how SSE-$S_+$ refines SE-$S_+$, we would like to understand how SSE-$S$ refines SE-$S$.

It was known that SE-$S$ implies SSE-$S$ if

\[ S = \mathbb{Z} \text{ (Williams, 70s)} \]
\[ S = \text{PID (Effros, 80s)} \]
\[ S = \text{Dedekind domain (B-Handelman, 90s)}. \]

That was it.
Definitions

\[ \text{GL}(S) = \text{group of } \mathbb{N} \times \mathbb{N} \text{ matrices } \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \]
with \( U \) finite invertible.

\[ \text{EL}(S) = \text{subgroup generated by basic elementary matrices } E \]
\( (E = I \text{ except perhaps in one offdiagonal entry}) \)

\[ \text{EL}(S) = \text{commutator subgroup} \]

\[ K_1(S) = \text{GL}(S) / \text{EL}(S) \]

The central connection for clarifying SSE-\( S \) is ...

THEOREM (B-Schmieding)
Suppose $A, B$ are matrices over $S$. TFAE.

(1) $A$ and $B$ are SSE over $S$.

(2) There are $E, F$ in $\text{El}(S[t])$ such that $E(I - tA)F = (I - tB)$.

The finite matrices $I - tA, I - tB$ are embedded as the upper left corners of matrices with all other entries zero (and identified with these infinite matrices).

This grows out of work by Shannon, BGMY, Wagoner, Kim-Roush-Wagoner, B-Sullivan.

The theorem above leads to ...
THEOREM (B-Schmieding)
Let $A$ be a square matrix over $S$.

(I) If $B$ is SE over $S$ to $A$, then there is a nilpotent matrix $N$ such that

$$\begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix}$$

is SSE over $S$ to $B$.

(II) The map

$$\begin{pmatrix} A & 0 \\ 0 & N \end{pmatrix} \to I - tN$$

induces a bijection from the set of SSE classes of matrices SE over $S$ to $A$ to the abelian group $NK_1(S)/H_A$. 
The group $NK_1(S)$ is an important group in the algebraic K-theory of the ring $S$. It is the kernel of the map

$$K_1(S[t]) \to K_1(S)$$

induced by $t \mapsto 0$.

The group $H_A$ is the set of elements in $K_1(S)$ containing a matrix $U$ such that there is $E$ in $\text{El}(S)$ such that

$$U(I - tA)E = I - tA.$$
What about this group

\[ NK_1(S)/H_A \]

which captures the refinement of SE-\( S \) by SSE-\( S \)?

\( NK_1(S) \) if nontrivial is not finitely generated (Farrell 1977).

\( H_A = 0 \) if \( A \) is nilpotent or \( S \) is commutative.

Any consequences of Theorem?
Known fact: for $S = ZG$ with $G = \mathbb{Z}/n\mathbb{Z}$: 
$NK_1(S) = 0$ iff $n$ is squarefree.

For the not-squarefree case: we expect this will let us refute a working conjecture of Bill Parry on the classification of skew products of mixing SFTs by finite groups.

For a huge class of rings, we now know SE-$S$ implies SSE-$S$. This includes $ZG$ with $G = \mathbb{Z}^n$.

THM. Suppose $A$ and $B$ are matrices over a dense subring $S$ of the reals, with $A$ primitive and $B$ SE over $S$ to $A$, with trace$(A) > 0$. Then $B$ is SSE over $S$ to a primitive matrix.

(The “Generalized Spectral Conjecture” of B-Handelman is reduced to realization by any element of a shift equivalence class.)
In “Path Methods for strong shift equivalence of positive matrices” (B-Kim-Roush 2013), the constructions of certain SSEs of positive matrices $A, B$ over $S$ a dense subring of $R$ depended on an assumption $A, B$ SSE over $S$ (not just SE). We now know this is not an artifact of a deficient proof. E.g., $S = \mathbb{Q}[\pi^2, \pi^3, e, e^{-1}]$ has $NK_1(S)$ nontrivial.

In (B-Kim-Roush 2013), a 3-step program was proposed for understanding SSE-$S_+$ of positive trace matrices over $S$ a dense subring of $\mathbb{R}$. One step was to understand the refinement of SE by SSE over $S$.

In this work, we found a characterization of equivalence in the Bass group $\text{Nil}_0(S)$ which (so far?) we have not found in the literature.
The connections involved in these results may lead to ideas useful for understanding the $\mathbb{Z}_+^n$ case of SSE. This suggestion is perhaps not so wild as it might appear.

As Sinai replied, when asked after a talk whether he thought his probabilistic approach to the Mobius subshift could lead to a proof of the Riemann Hypothesis:
The situation is not hopeless.