Diffusion Limited Aggregation Forest

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Joint work with Noam Berger (HUJI) and Eviatar B. Procaccia (WIZ)

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- The internal diffusion limited aggregation (IDLA) was first proposed by Meakin and Deutch (1986) to model industrial chemical processes like electropolishing, corrosion and etching.

Figure: IDLA of $n = 4 \times 10^5$ points. (Jerison, Levine and Sheffield 2012)

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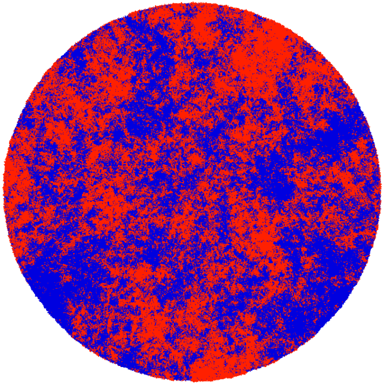

Figure: IDLA of $n = 4 \times 10^5$ points. (Jerison, Levine and Sheffield 2012)

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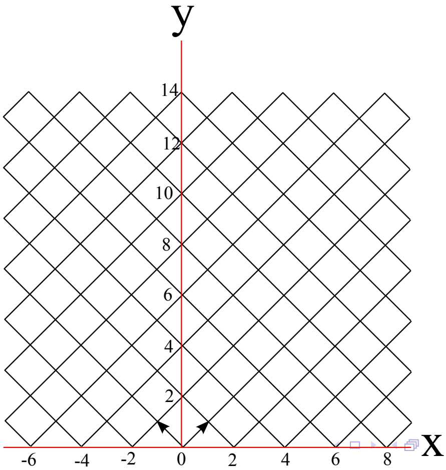
the model

- Consider the upper half of the $\mathbb{Z}^2$ rotated lattice.
- With every even vertex on the x axis associate an independent poisson clock.



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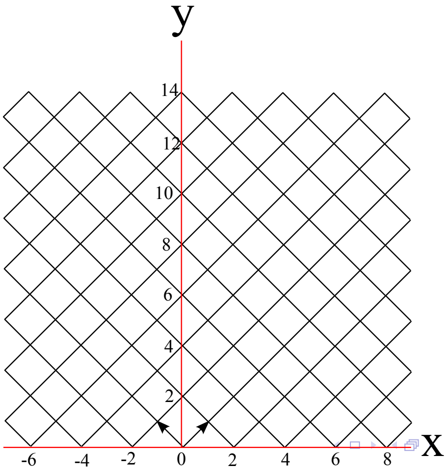
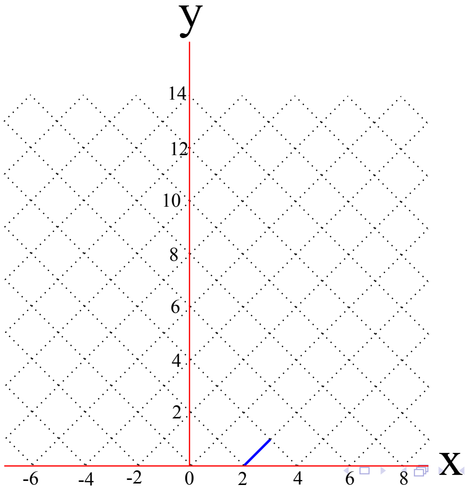


- When a clock of some vertex rings, start a random walk from the vertex until reaching the cluster's boundary.
- If the walk tries to move to an adjacent tree or to close a loop, it dies.



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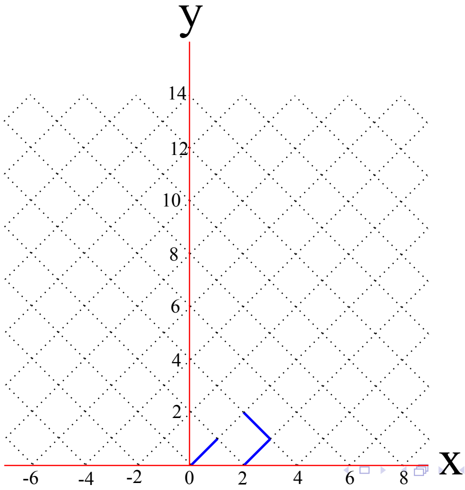

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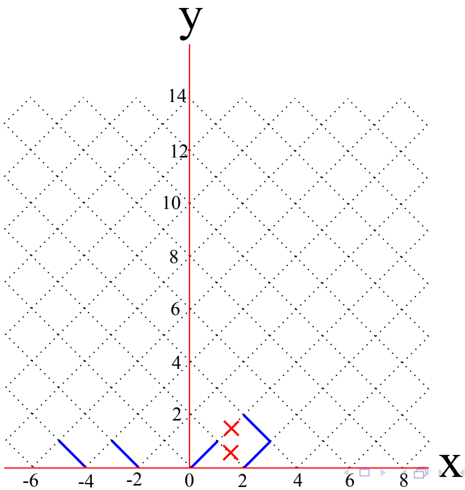

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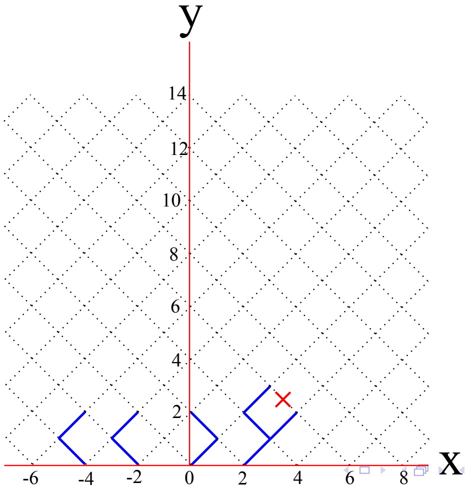

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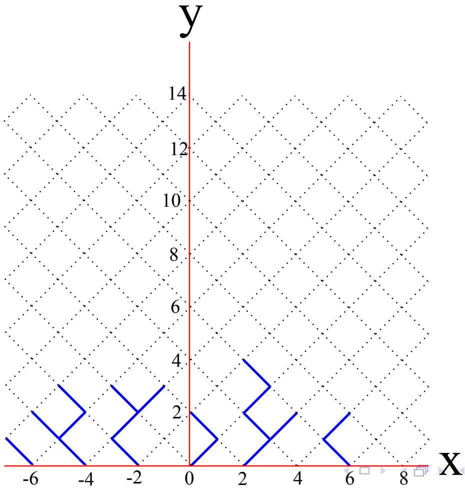
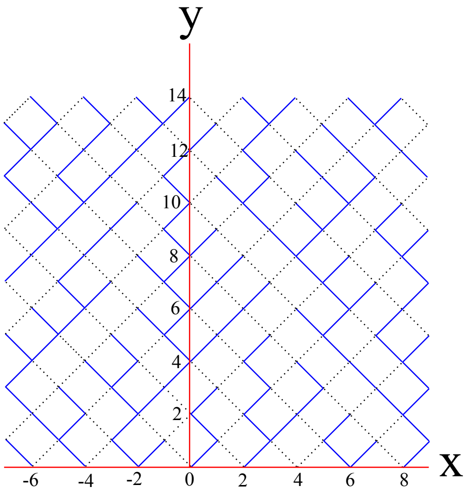
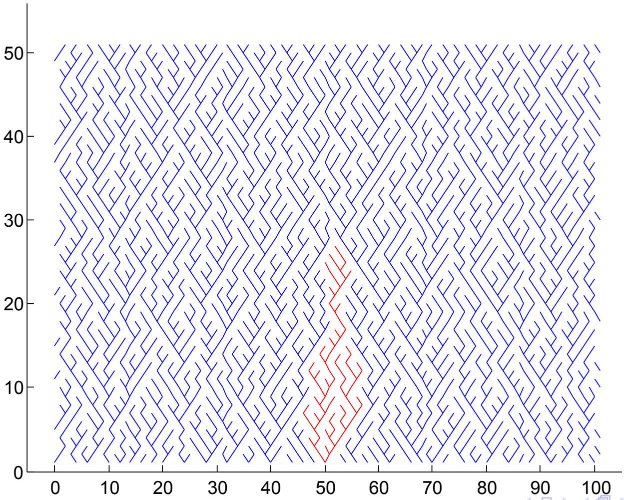


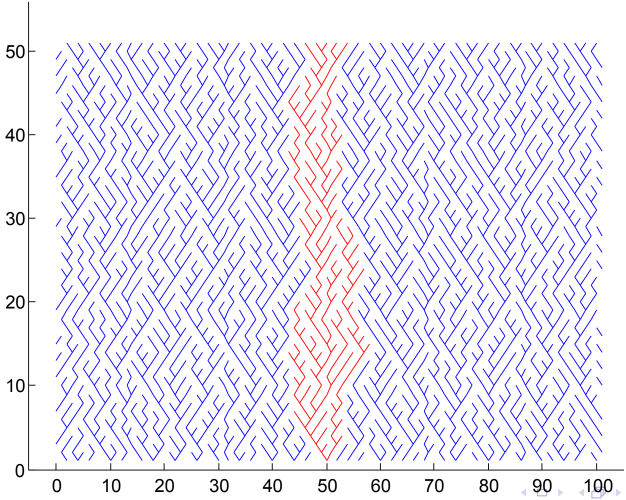
- This way we get a forest that will eventually reach every vertex.



Question: Are all the trees finite?



Question: Or do infinite trees exist?



First Passage percolation

- First passage percolation (FPP) is a random metric on a graph.
- Each edge e is given a weight $w(e)$.
- The distance between two vertices is the minimal sum of weights over paths connecting the vertices.

$$d_\omega(x, y) = \min_{\gamma: x \to y} \sum_{e \in \gamma} w(e)$$

- note that this definition generalizes in an obvious way to a distance between sets.

$$d_\omega(A, B) = \min_{\substack{\gamma: x \to y \\ x \in A, y \in B}} \sum_{e \in \gamma} w(e)$$

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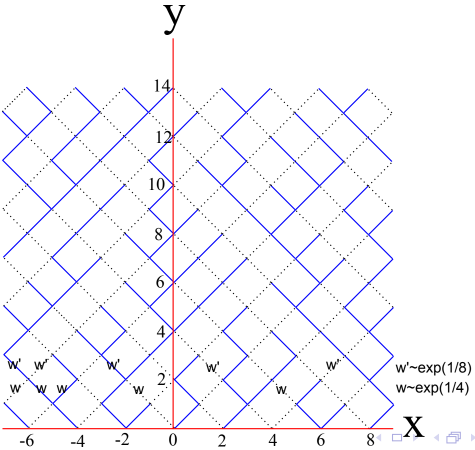
the coupling

We couple the process to a FPP process with weights $\sim \exp(\frac{1}{2^n})$ on level n, denoting by $\hat{T}$ the measure on trees which are the union of geodesics from all vertices to the x axis.



Lemma 1

There exists a coupling measure Q such that

$$Q(\forall x, T(x) = \hat{T}(x)) = 1$$

Proof sketch.

For each monotone path $\gamma \subseteq \{\hat{T}(x) \cup \partial \hat{T}(x)\}$, $\gamma = (e_1, \ldots, e_{l(\gamma)})$ from x we assign rings:

- if $\gamma \subset \hat{T}(x)$ we assign the ring $\sum_{i=1}^{l(\gamma)} \omega(e_i)$, and the path of the particle will be γ.
- if $\gamma \not\subseteq \hat{T}(x)$ we assign the ring sequence $\sum_{i=1}^{l(\gamma)} \omega(e_i)$, $\sum_{i=1}^{l(\gamma)} \omega(e_i) + Poisson(e_{l(\gamma)})$, for each ring in this sequence the particle will be assigned the path γ.

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Note that in the second case, all the particles will vanish

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Theorem 3.1

All the trees are finite almost surely.

Proof sketch.

- Assume an infinite tree exists. Denote $p = \mathbb{P}(|T(0)| = \infty)$
- by translational invariance we have

$$\mathbb{E}[|T^m(0)||T(0) = \infty] \leq \frac{1}{p}$$

- we try to kill the tree at a sequence of levels where the tree has a fixed diameter.
- this is a sequence of independent, finite events with positive probability
- a Borel-Cantelli argument gives a contradiction.

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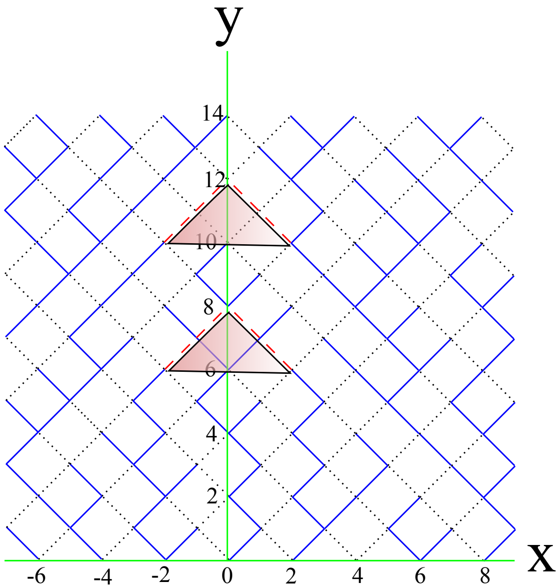
y
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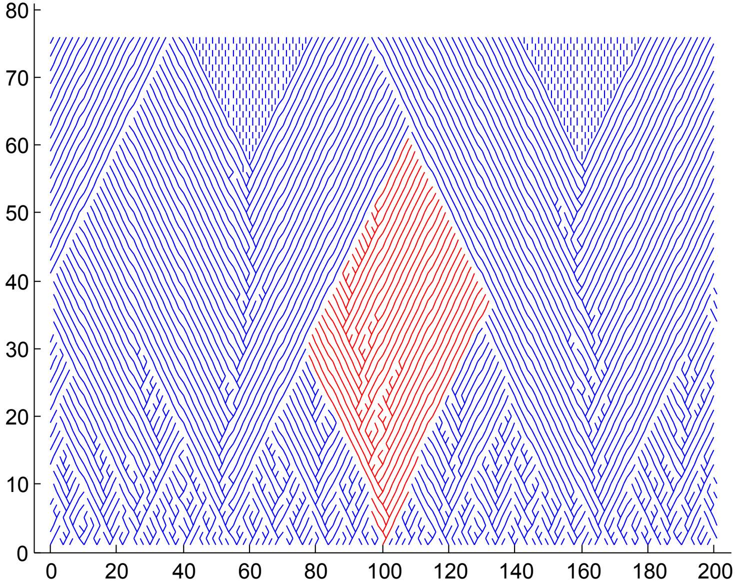
FPP with decreasing weights

Theorem 3.2

All the trees with edge weights $\exp(2^n)$ *are finite almost surely.*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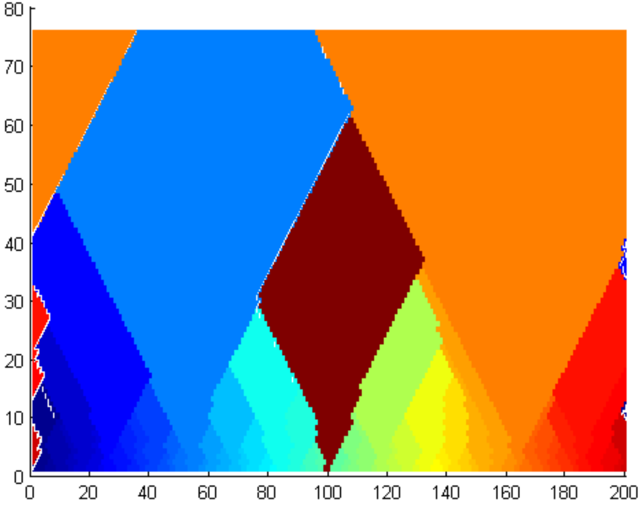
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Questions?