High Dimensional Expanders

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A finite graph $X = (V, E)$ is ε-expander $(0 < \varepsilon \in \mathbb{R})$ if

$$h(X) \geq \varepsilon \;\text{ where}$$

$$h(X) = \text{The Cheeger constant } = \min_{\phi \neq A \subset V} \frac{|E(A, \bar{A})|}{\min(|A|), |\bar{A}|)}$$

- ε-expander is connected, in fact "strongly connected".

The above definition is the "right one" for k-regular graphs, k-fixed.
One which works well also for general graphs:

$$\text{Discrepancy } = \mathrm{Dis}(X) < \varepsilon$$

$$\text{where} \quad \mathrm{Dis}(X) = \min_{0 \neq A \subset V} \left| \frac{E(A, \bar{A})}{|E|} - |\frac{A}{V}| \cdot |\frac{\bar{A}}{V}| \right|$$

i.e., how far X is from random.

So expanders are "approximately random" and this is another reason for their many applications.

Examples

Some history :

- Random k-regular graphs. (Pinsker 1973)
- Constructive method using Kazhdan Property (T). (Margulis 1975)
- Ramanujan graphs (Lubotzky-Phillips-Sarnak, Margulis 1988)
- The Zig-Zag product (Reingold-Vadhan-Wigderson 2002)
- Interlacing polynomials (Marcus-Spielman-Srivastava 2013)

In most applications (but not all) one wants k-regular graphs, k fixed, $\varepsilon > 0$ fixed and $|X| \to \infty$.

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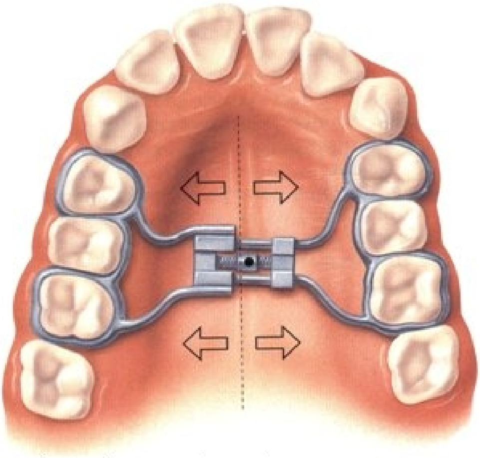
Expanders are extremely important in CS, combinatorics and even in pure mathematics.

See A. Lubotzky, *Expanders in pure and applied mathematics*, Bull AMS 2012.

... over 4,000,000 hits in google, but most of them are ...



How to define "high dim. expanders"?

Several approaches:

Let's start with Gromov approach, but first another story:

Theorem (Boros-Füredi '84)

Given a set $P \subseteq \mathbb{R}^2$, with $|P| = n$, $\exists z \in \mathbb{R}^2$ which is covered by $(\frac{2}{9} - o(1))\binom{n}{3}$ of the $\binom{n}{3}$ triangles determined by P.

Remark:

$\frac{2}{9}$ is optimal.

Theorem (Barany)

$\forall d \geq 2,\ \exists c_d > 0$ s.t. $\forall P \subset \mathbb{R}^d$ with $|P| = n$, $\exists z \in \mathbb{R}^d$ which is covered by at least $c_d\binom{n}{d+1}$ of the d-simplices determined by P.

Gromov proved the following remarkable result; but first a definition:

Definition

Let X be a d-dimensional simplicial complex. We say that X has the *geometric* (resp. *topological*) ε*-overlapping property* if:

$$\forall f : X(0) \to \mathbb{R}^d \text{ and } \forall \tilde{f} \text{ affine (resp. continuous) extension } \tilde{f} : X \to \mathbb{R}^d,$$

there exists $z \in \mathbb{R}^d$ which is covered by ε-fraction of the images of $X(d)$ ($= d$-dim simplices).

Barany's Theorem means: the complete d-dim complex $\triangle_n^{(d)}$ on n vertices has the *geometric* ε-overlapping property.

Theorem (Gromov 2010)

$\triangle_n^{(d)}$-(d fixed, $n \to \infty$) also has the c_d-*topological* overlapping property.

Think about the case $d = 2$ to see how non-trivial is this theorem and even somewhat counter-intuitive.

What does this have to do with expanders?

Look at $d = 1$ and assume $X = (V, E)$ is ε-expander.

Let $f : X(0) = V \to \mathbb{R}^1 = \mathbb{R}$.

Take $z \in \mathbb{R}$ s.t. $1/2$ of the images are below it and $1/2$ above.

If $A =$ the vertices above, then all the edges of $E(A, \bar{A})$ pass through z.

As X is an expander $E(A, \bar{A})$ is "large" and we have *topological* overlapping.

Definition

A family of d-dim s.c.'s is **geometric (resp. topological) expanders** if all have the ε-geometric (resp. topological) overlapping property for the same $\varepsilon > 0$.

Remark: Expander is stronger than top. overlapping.

While it is trivial to prove that the complete graphs are expanders, the higher dim case of complete complexes (i.e. Gromov's theorem) is highly non-trivial.

Various methods show existence of bounded degree expander graphs: Random, Kazhdan property (T), Ramanujan conjecture/graphs, Zig-Zag ...

Are there bounded degree geometric/topological expanders?

Geometric overlapping

Theorem (Fox-Gromov-Lafforgue-Naor-Pach 2013)

$\forall d, \exists$ bounded degree (i.e. every vertex is contained in a bounded number of simplices) simplicial complexes of $\dim d$ with geometric overlapping.

Two methods of proof:

- Random
- Ramanujan complexes

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Topological overlapping; very partial results

Theorem (Lubotzky-Meshulam 2014)

W.r.t. a suitable model of 2-dim random simplices, with full 1-skeleton (so *not* bounded degree vertices) with *bounded edge degree*, almost every such 2-complex is a topological expander.

- The model is based on "Latin squares"

"Theorem" (Kaufman-Kazhdan-Lubotzky 2014)

The 2-skeletons of suitable 3-dim Ramanujan complexes are simplicial complexes of bounded degree with the topological overlapping property.

- Needs either Serre conjecture on the congruence subgroup property or an extension of Gromov (by T. Kaufman and U. Wagner).
- Gives bounds on the cohomological systole (mod 2) which are of value for quantum error correcting codes.