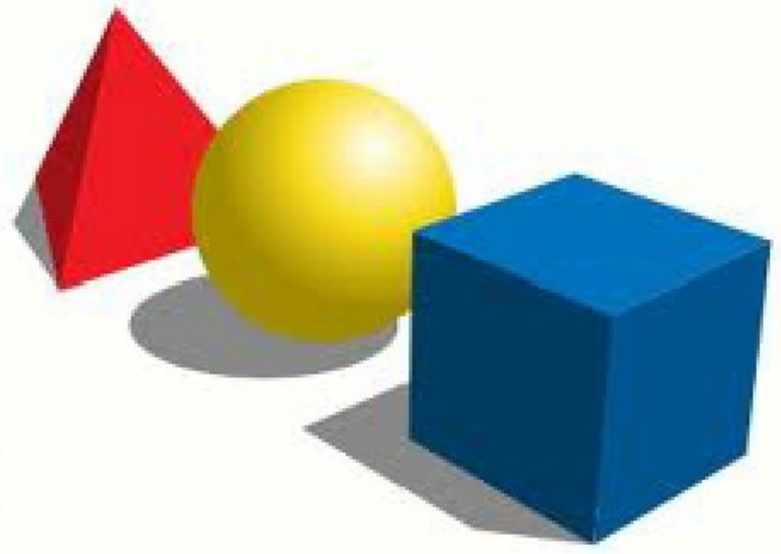
Numbers and shapes

Henri Darmon

Hugh C. Morris Lecture

Calgary, November 1, 2012



Diophantine equations

A *diophantine equation* is a system of polynomial equations, in which the variables are only allowed to take *integer values*. (One sometimes considers also *rational solutions*.)

Solutions to differential equations typically admit natural interpretations in terms of systems (arising in physics, biology, finance, ...) which one is trying to model.

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An equation at random

Is the equation

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Fermat's Last Theorem

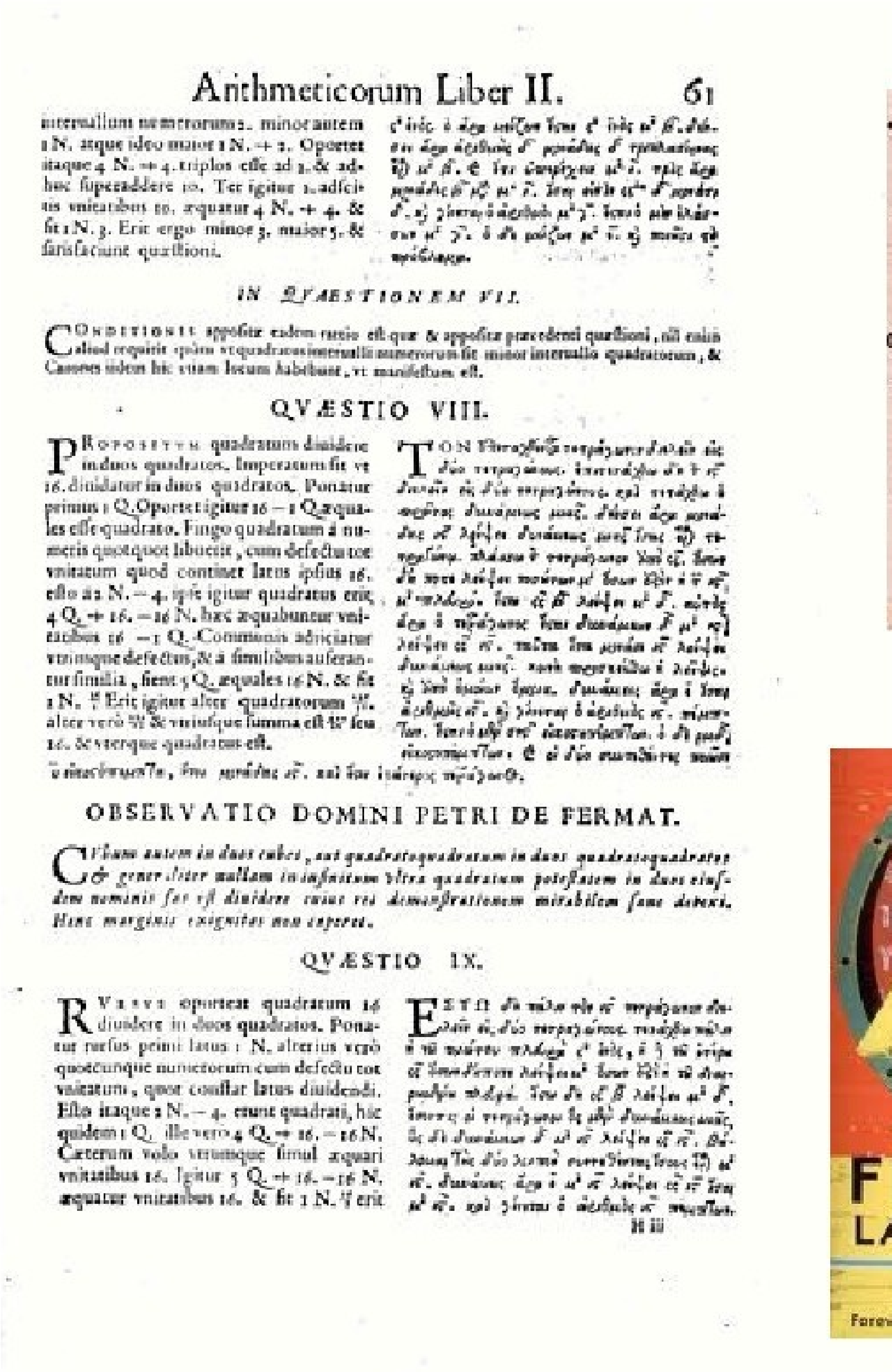
Theorem (Wiles)

If $n \geq 3$, the equation

$$x^n + y^n = z^n$$

has no solutions $(x, y, z) \in \mathbb{Z}^3$ with $xyz \neq 0$.

This equation is certainly interesting!



Arithmeticorum Liber II. 61

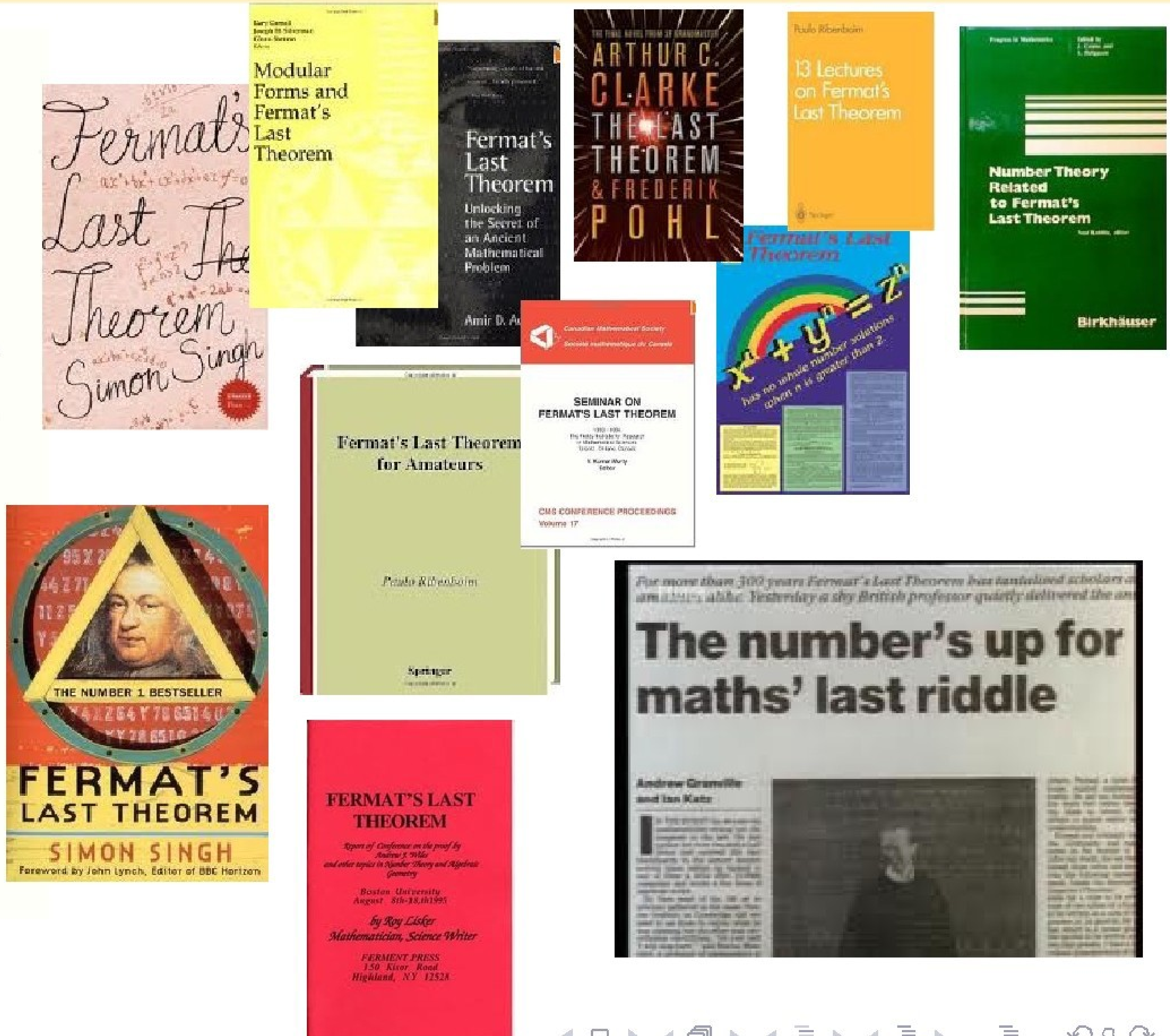
IN QVAESTIONEM VII.

QVAESTIO VIII.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVAESTIO IX.



Frey's equation

A putative solution $a^p + b^p = c^p$ to the Fermat equation gives rise to the *auxiliary equation* in two variables x, y:

$$E_{a,b,c} : y^2 = x(x - a^p)(x + b^p).$$

One is *reduced* — in a way which will seem artificial *a priori* — to understanding certain *cubic equations in two variables* well enough to show that the one above *cannot exist*.

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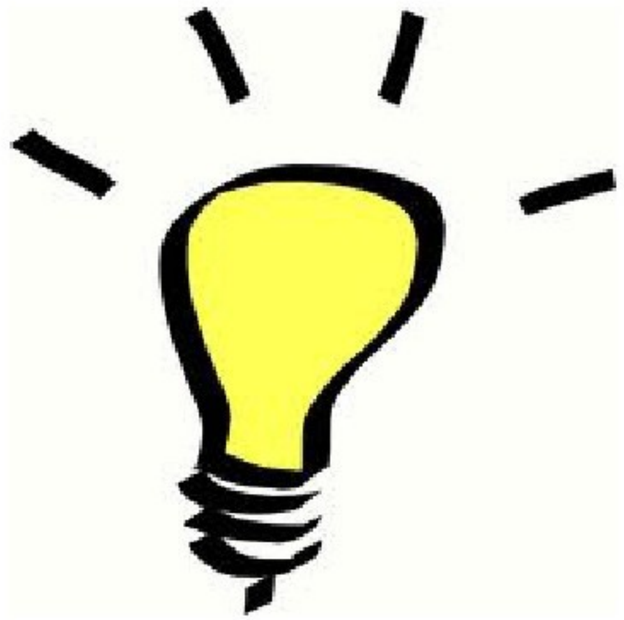

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What is special about Frey's equation?

Give any prime ℓ, consider the *reduction* of $E_{a,b,c}$ modulo ℓ.

Proposition

Frey's elliptic curve $E_{a,b,c}$ has good reduction*–i.e., no singular points — modulo ℓ if and only if $\ell \nmid abc$.*

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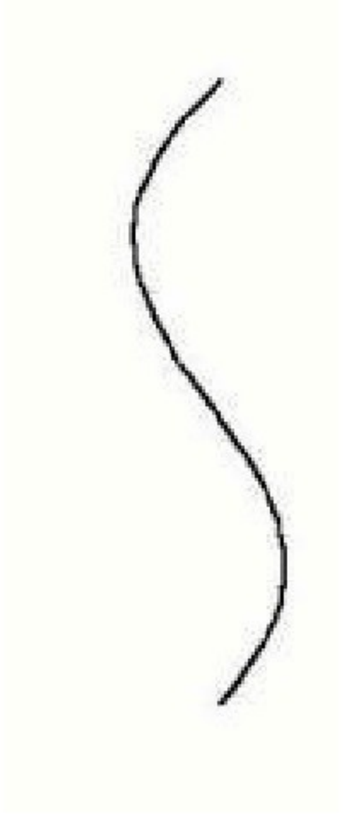

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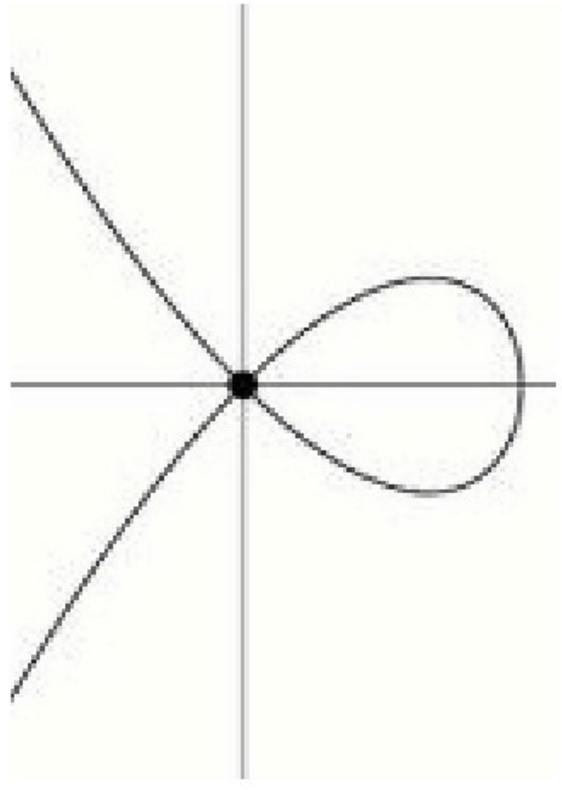

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Elliptic curves

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An *elliptic curve* over $\mathbb{Q}$ is an equation of the form

$$y^2 = x^3 + ax + b,$$

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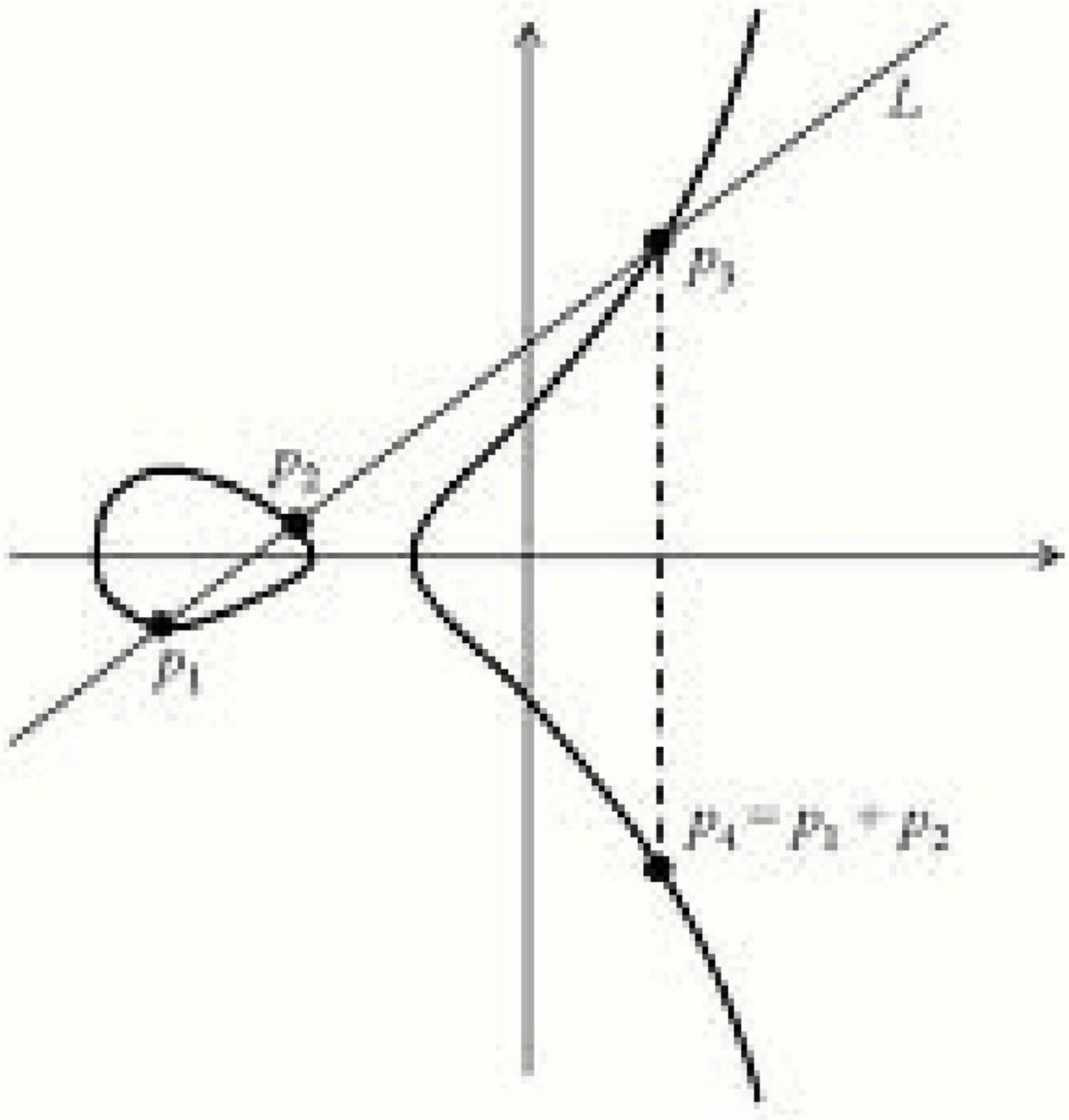
The addition law (over $\mathbb{R}$)



The addition law over $\mathbb{Q}$

The addition law makes it possible to generate *new rational solutions* from previously known ones: if (x_1, y_1) and (x_2, y_2) are rational solutions, then the pair (x_3, y_3) given by

$$\begin{aligned} x_3 &= \left(\frac{y_1 - y_2}{x_1 - x_2}\right)^2 - x_1 - x_2; \\ y_3 &= y_1 + \left(\frac{y_1 - y_2}{x_1 - x_2}\right)(x_3 - x_1) \end{aligned}$$

is yet another rational solution.

Consequence: The $\mathbb{F}_p$-vector space

$$E[p] := E(\bar{\mathbb{Q}})[p] = \{Q \in E(\bar{\mathbb{Q}}) \quad pQ = 0\}$$

is equipped with a natural *linear* action of $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

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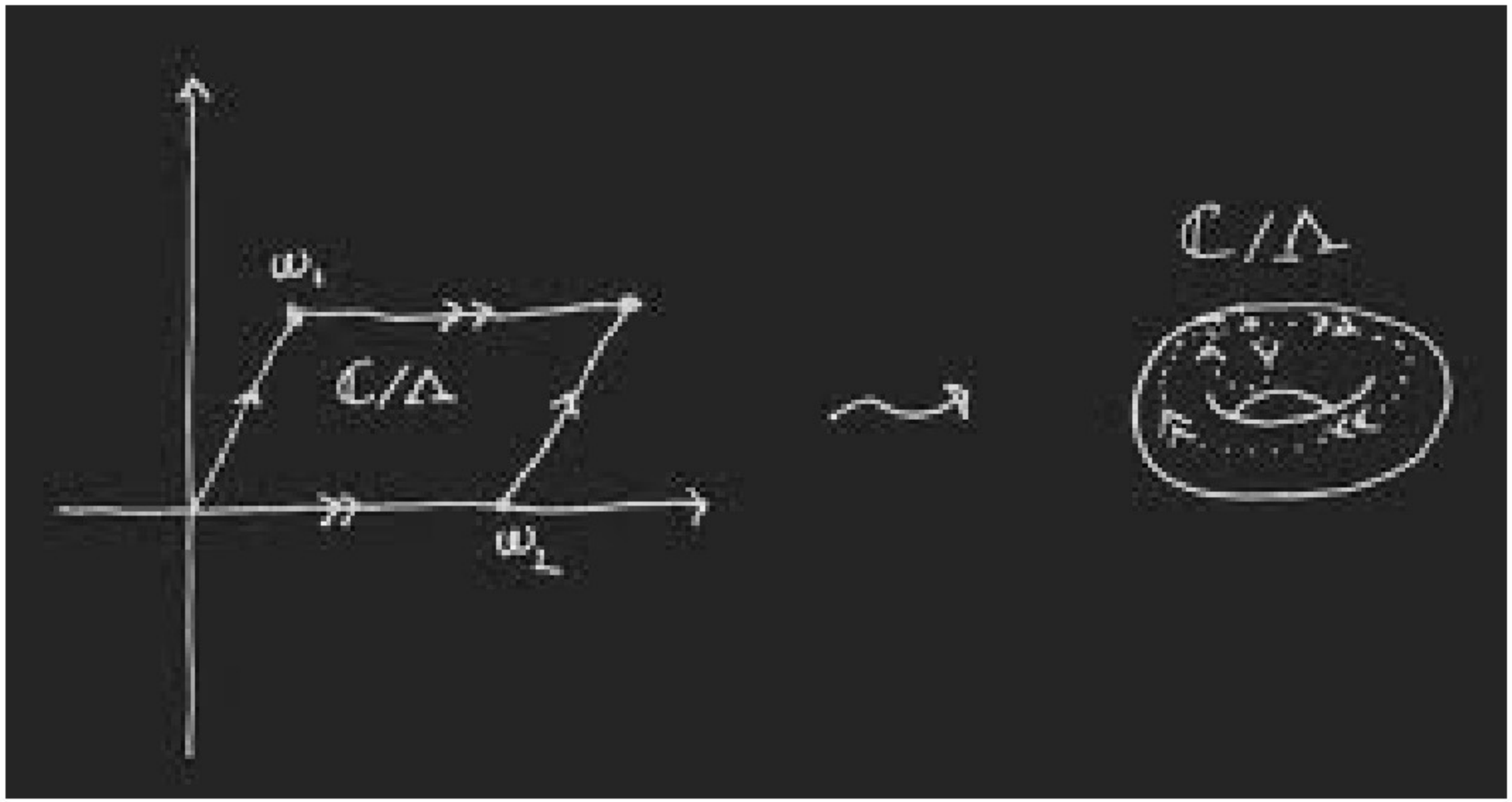
The addition law over $\mathbb{C}$

Weierstrass theory identifies the abelian group $E(\mathbb{C})$ with $\mathbb{C}/\Lambda$ where Λ is a *lattice*.



Mod p Galois representations

A consequence of the complex description of the addition law is that

$$E(\mathbb{C})[p] = \frac{1}{p}\Lambda/\Lambda \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.$$

The same is true when $\mathbb{C}$ is replaced by $\bar{\mathbb{Q}}$. Hence

$$E[p] := E(\bar{\mathbb{Q}})[p]$$

is a *two-dimensional vector space* over the field with p elements. Recall from the previous slide that it is equipped with a continuous linear action of $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

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Proposition (Frey)

The representation $E_{a,b,c}[p]$ behaves as if *$E_{a,b,c}$ had non-singular reduction modulo* all primes *$\ell \neq 2$, including the odd primes dividing abc.* *(More precisely, it is* unramified *at all primes $\ell \neq 2, p$ and is (*cristalline*) at p.)*

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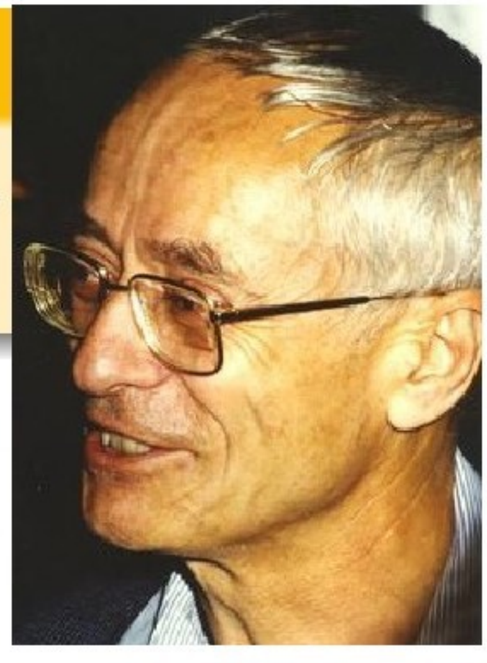

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Solutions of FLT $\Rightarrow$ Frey curves

$\Rightarrow$ Galois representations

To a solution (a, b, c) to $x^p + y^p = z^p$, we associate the *Frey elliptic curve* $E_{a,b,c}$, an equation whose arithmetic complexity depends on (a, b, c).

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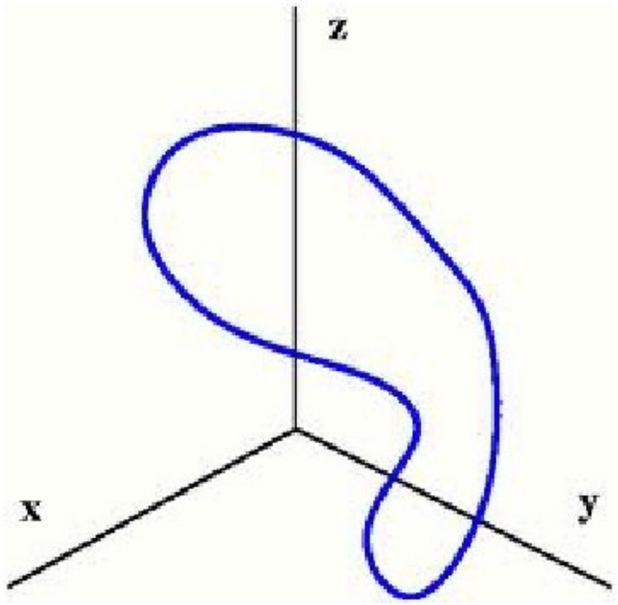
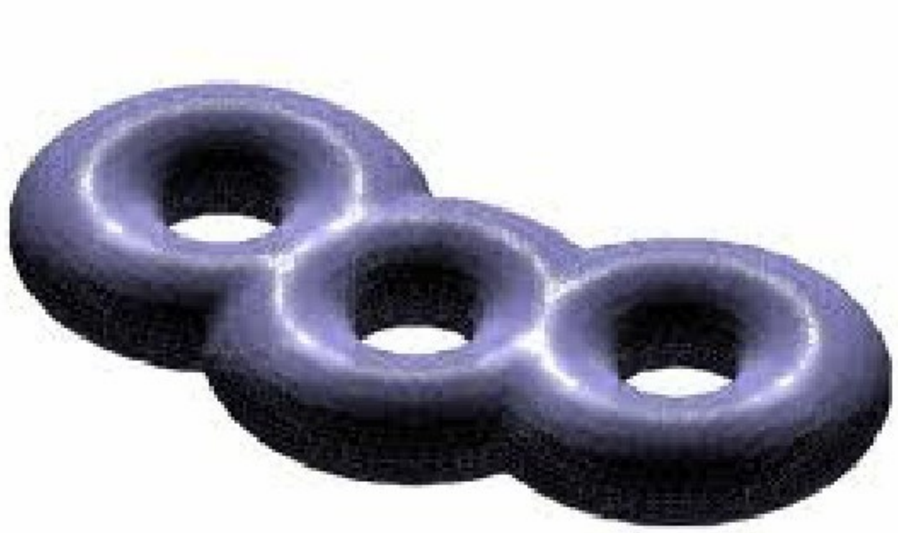
Curves

A *curve* over $\mathbb{Q}$ is a diophantine equation X whose underlying solution set over $\mathbb{R}$ is one-dimensional–and hence, whose underlying solution set over $\mathbb{C}$ is a *Riemann surface*. E.g., $f(x, y) = 0$.





Faltings' theorem, a.k.a. the Mordell conjecture

Theorem (Faltings)

Let X be a curve. Then $X(K)$ is finite for all number fields K if and only if genus$(X(\mathbb{C})) \geq 2$.

Parshin's "trick"

Parshin: To a rational solution of X—i.e., a point $a \in X(\mathbb{Q})$—one can associate an *auxiliary curve* X_a with the following properties:

1. The assignment $a \mapsto X_a$ is injective;

2. The genus $g > 1$ of X_a depends only on X and not on a;

3. The *arithmetic complexity* of X_a—i.e., the set of primes modulo which is acquires singular reduction—depends only on X as well.

Mordell's conjecture is now a consequence of

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$$J_a := \mathrm{Jac}(X_a) = \frac{\text{degree 0 divisors on } X_a}{\text{principal divisors on } X_a}.$$

It is an *abelian variety* of dimension g over $\mathbb{Q}$: a higher dimensional analogue of an elliptic curve.

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As abelian groups,

$$J_a[p^\infty] := \varprojlim J_a[p^n] \simeq \mathbb{Z}_p^{2g}.$$

$$V_{a,p} := J_a[p^\infty] \otimes \mathbb{Q}_p \simeq \mathbb{Q}_p^{2g}.$$

These groups and vector spaces are equipped with a continuous linear action of $G_{\mathbb{Q}}$.

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As abelian groups,

$$J_a[p^\infty] := \varprojlim J_a[p^n] \simeq \mathbb{Z}_p^{2g}.$$

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Cohomological interpretation

The representations $J_a[p^\infty]$ and $V_{a,p}$ admit a cohomological interpretation.

If $g = 1$, so that $X(\mathbb{C}) = \mathbb{C}/\Lambda$, then

$$J_a[p^\infty] = \varinjlim \frac{1}{p^n}\Lambda/\Lambda = \varprojlim \Lambda/p^n\Lambda = \Lambda \otimes \mathbb{Z}_p = H_1(E(\mathbb{C}), \mathbb{Z}_p).$$

For arbitrary g,

$$H_1(X, \mathbb{Q}_p) := \mathrm{Jac}(X)[p^\infty] \otimes \mathbb{Q}_p$$

is equipped with a continuous linear action of $G_{\mathbb{Q}}$.

Hence $H_1(-, \mathbb{Q}_p)$ is a functor from the category of curves over $\mathbb{Q}$ to the category of continuous linear representations of $G_{\mathbb{Q}}$.

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Etale cohomology

The extra structure on $H_1(X, \mathbb{Q}_p)$ given by the $G_{\mathbb{Q}}$-action carries a great deal of information about X.

1. For all $\ell \neq p$ at which X has good reduction, the representation $H_1(X, \mathbb{Q}_p)$ is unramified at ℓ, and the characteristic polynomial $P_\ell(x) \in \mathbb{Q}_p[x]$ of the frobenius element σ_ℓ at ℓ belongs to $\mathbb{Z}[x]$ and is independent of p.
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$$\#X(\mathbb{F}_\ell) = \ell + 1 - \mathrm{trace}(\sigma_\ell|_{H_1(X,\mathbb{Q}_p)}).$$

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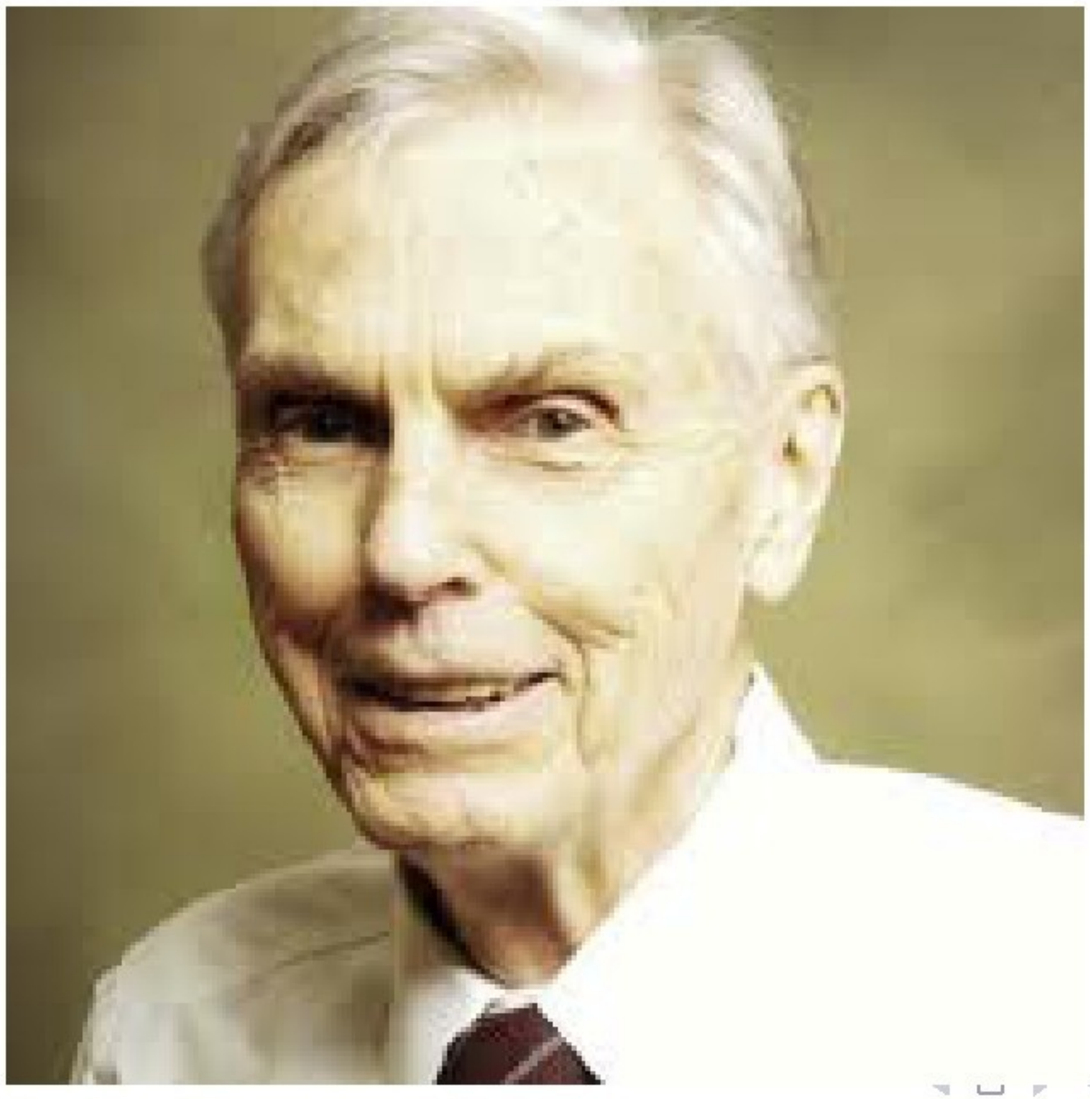
The Tate conjecture

Conjecture (Tate)

If $H_1(X_1, \mathbb{Q}_p) \simeq H_1(X_2, \mathbb{Q}_p)$, *then there is a finite morphism* $Jac(X_1) \longrightarrow Jac(X_2)$.



Faltings' finiteness theorems

Theorem (Faltings)

1. *The Tate conjecture is true, and there are finitely many abelian varieties varieties in a given isogeny class. In particular, the assignment $a \mapsto H_1(X_a, \mathbb{Q}_p)$ is finite-to-one.*
2. *The representation $H_1(X, \mathbb{Q}_p)$ is semi-simple, and there are finitely many rational semi-simple p-adic representations of $G_{\mathbb{Q}}$ with "bounded arithmetic complexity", up to isomorphism.*

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Rational points $a \in X(\mathbb{Q})$ $\Rightarrow$ Curves X_a

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Constructing points

The results described above illustrate settings where one tries to bound the set of solutions to Diophantine equations *from above*,

1. by showing there are no solutions at all, as in Wiles' proof of Fermat's Last Theorem;
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What about the *construction* of explicit solutions?

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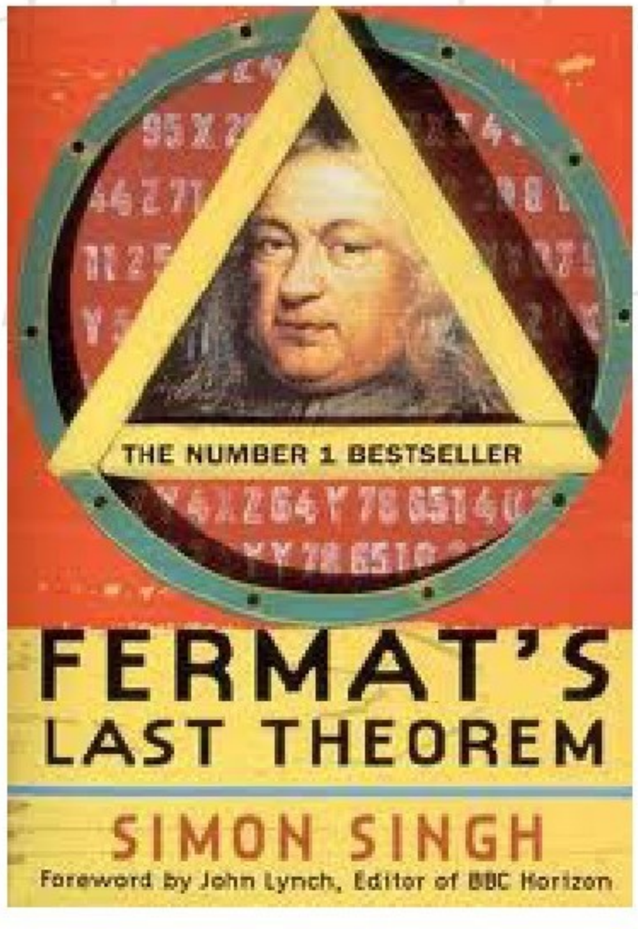

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Elliptic curves

We consider only the simplest non-trivial setting: an elliptic curve E over $\mathbb{Q}$ (with good reduction outside a finite set S of primes).

Given $a \in E(\mathbb{Q})$, the most natural variety one can associate to it is the *open curve*

$$E_a := E - \{0, a\}.$$

Its homology and cohomology with coefficients in L fit into functorial ("excision") exact sequences

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The many faces of cohomology theories

In (arithmetic) algebraic geometry, cohomology functors come in many guises, and are equipped with a plethora of extra structures:

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Mixed Hodge structures

When X is an (open) curve, the data

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is a prototypical example of an *integral (mixed) Hodge structure*.

Proposition

If E is an elliptic curve, and $a \in E(\mathbb{C})$, the mixed Hodge structure $H^1_{\mathrm{dR}}(E_a)$—more precisely, its class in $\mathrm{Ext}^1_{MHS}(\mathbb{C}(-1), H^1_{\mathrm{dR}}(E))$—determines the complex point a completely.

Indeed, the class of E_a is encoded in the complex line integral

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Conjecture (Shafarevich, Tate)

If E is an elliptic curve, and ξ is a class in $\mathrm{Ext}^1_{Rep_{G_{\mathbb{Q}}}}(\mathbb{Q}_p(-1), H^1_{\mathrm{et}}(E, \mathbb{Q}_p))$ satisfying suitable explicit "local conditions", then ξ is proportional to the class of E_a, for some $a \in E(\mathbb{Q})$.

Extensions of p-adic Galois representations

Given $a \in E(\mathbb{Q})$, the mixed Hodge structure $H^1_{\mathrm{dR}}(E_a)$ has a counterpart in the realm of p-adic Galois representations: the *étale cohomology group* $H^1_{\mathrm{et}}(E_a, \mathbb{Q}_p)$ which fits into an exact sequence

$$0 \longrightarrow H^1_{\mathrm{et}}(E, \mathbb{Q}_p) \longrightarrow H^1_{\mathrm{et}}(E_a, \mathbb{Q}_p) \longrightarrow \mathbb{Q}_p(-1) \longrightarrow 0.$$

Conjecture (Shafarevich, Tate)

If E is an elliptic curve, and ξ is a class in $\mathrm{Ext}^1_{Rep_{G_{\mathbb{Q}}}}(\mathbb{Q}_p(-1), H^1_{\mathrm{et}}(E, \mathbb{Q}_p))$ satisfying suitable explicit "local conditions", then ξ is proportional to the class of E_a, for some $a \in E(\mathbb{Q})$.

Conclusion

To construct points on elliptic curves, it "suffices" (modulo some deep conjectures!) to construct suitable

1. extensions of $\mathbb{C}(-1)$ by $H^1_{\mathrm{dR}}(E)$, in the category of *mixed Hodge structures*.
2. extensions of $\mathbb{Q}_p(-1)$ by $H^1_{\mathrm{et}}(E, \mathbb{Q}_p)$, in the category of continuous p-adic representations of $G_{\mathbb{Q}}$.

The first approach is good for practical numerical computations, while the second tends to be better for theoretical applications.

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Modular curves

The Poincaré upper half-plane $\mathcal{H}$ is acted on discretely by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) \text{ such that } N|c \right\}.$$

The algebraic curve $X_0(N)$ satisfying

$$X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash(\mathcal{H} \cup \mathbb{P}_1(\mathbb{Q})),$$

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Wiles' theorem, revisited

Theorem (Wiles)

The p-adic representation $H^1_{\text{et}}(E, \mathbb{Q}_p)$ is a constitutent of $H^1_{\text{et}}(X_0(N), \mathbb{Q}_p)$ for a suitable integer N (the conductor *of E).*

Corollary (Faltings)

There is a non-constant morphism $X_0(N) \longrightarrow E$ of curves over $\mathbb{Q}$.

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Modularity of points on elliptic curves

A point $a \in E(\mathbb{Q})$ is said to be *modular* if $H^1(E_a)$ arises in $H^1(Y)$ for *some* open modular curve.

Theorem (Gross-Zagier)

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Shimura varieties

There is a rich fauna of higher-dimensional generalisations of modular curves: the *Shimura varieties* which play a key role in the arithmetic theory of automorphic forms. An *open Shimura variety* is the complement of a Shimura subvariety.

Definition

Let E be an elliptic curve over a number field K. A point $a \in E(K)$ is *modular* if there is an (open) Shimura variety Y and an inclusion $H^1(E_a) \subset H^*(Y)(j)$. (In the category of mixed Hodge structures, or of p-adic Galois representations.)

Question: What pairs $(E_{/K}, a)$ are captured by constructions of this type?

This interesting question is largely open.

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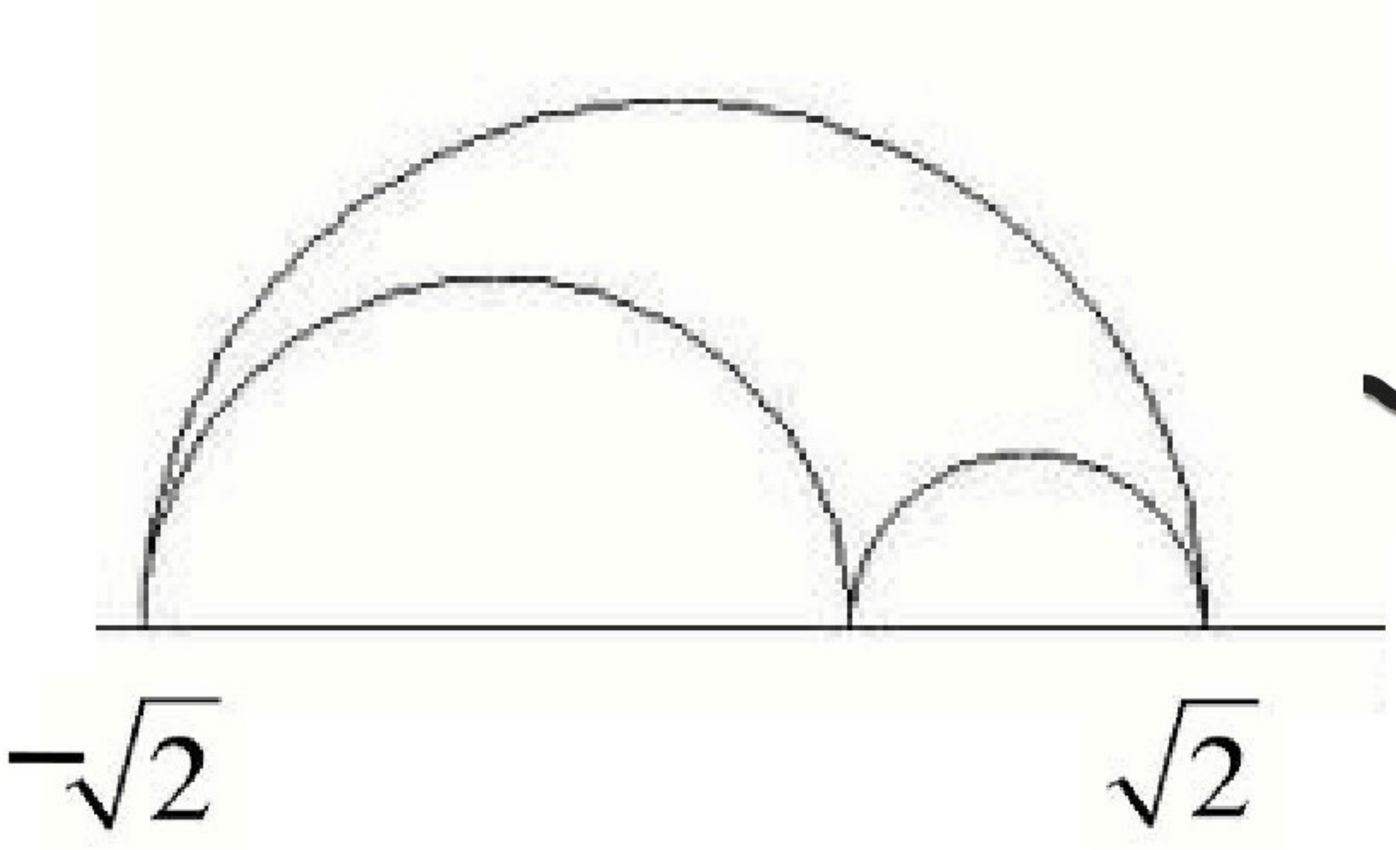

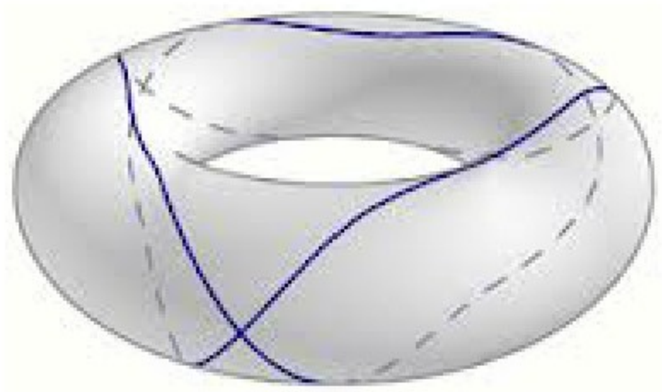

Stark-Heegner points

Nonetheless, such topological cycles can sometimes be used to construct explicit extensions of mixed Hodge structures (or p-adic variants thereof) which correspond, *conjecturally*, to global points on elliptic curves: the so-called *Stark-Heegner points*.

The phenomenon of Stark-Heegner points is quite mysterious and poorly understood — a better understanding of it would lead to fundamental insights into the arithmetic of elliptic curves, and to constructions of global points

1. defined over class fields of real quadratic fields;
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Stark-Heegner points

They have been intensely studied in the last years, both theoretically and experimentally, in Italy,

Massimo Bertolini

Matteo Longo

Marco Seveso

Stefano Vigni

Stark-Heegner points

In Spain,

Victor
Rotger

Xavier
Guitart

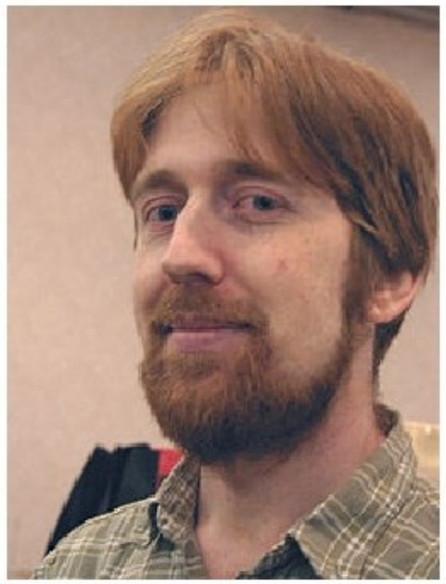
Marc
Masdeu

Stark-Heegner points

On the east coast,

Chung-Pang Mok

Robert Pollack



Adam Logan

Yu Zhao

Hugo Chapdelaine

Stark-Heegner points

.... and right here out West !

Matt Greenberg

Mak Trifkovic

A promising approach

One approach to understanding Stark-Heegner points is to try to construct the extension classes

$$\xi \in \mathrm{Ext}_{\mathrm{Rep}_{G_{\mathbb{Q}}}}(\mathbb{Q}_p(-1), H^1_{\mathrm{et}}(E, \mathbb{Q}_p))$$

that should arise from Stark-Heegner points, as *p-adic limits* of extension classes arising from a collection of open Shimura varieties.

A promising approach

Ongoing work—notably by Matt Greenberg and Marco Seveso—on p-adic L-functions for "Garrett Rankin triple products" offers hope that significant progress on these mysterious constructions will be achieved in the coming years.

Thank you for your attention.