Stochastic Equations of Super-Lévy Process with General Branching Mechanism

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- Further result: SPDE driven by $\alpha$-stable noise
Let \( \{X_t : t \geq 0\} \) be a binary branching super-Brownian motion (SBM). Then \( X_t(dx) = X_t(x)dx \) and the density is the unique positive weak solution to (Konno-Shiga (1988) and Reimers (1989)):

\[
\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + \sqrt{X_t(x)} \dot{W}_t(x), \quad t \geq 0, \quad x \in \mathbb{R},
\]

where \( \dot{W}_t(x) \) is the derivative of a space-time Gaussian white noise (GWN).

- The pathwise uniqueness for (1) is unknown.
  Progress: Perkins, Sturm, Mytnik, etc.

- Xiong (2012) studied the pathwise uniqueness to SPDE for the distribution function process of the SBM.
  Pathwise uniqueness to similar equation see Dawson and Li (2012).

- This talk is to generalize the result of Xiong (2012) to the super-Lévy process with general branching mechanism.
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- This talk is to generalize the result of Xiong (2012) to the super-Lévy process with general branching mechanism.
• $D(\mathbb{R}) := \{ f : f \text{ is bounded right continuous increasing and } f(-\infty) = 0 \}$.  

$M(\mathbb{R}) := \{ \text{finite Borel measures on } \mathbb{R} \}$.

There is a 1-1 correspondence between $D(\mathbb{R})$ and $M(\mathbb{R})$ assigning a measure to its distribution function. We endow $D(\mathbb{R})$ with the topology induced by this correspondence from the weak convergence topology of $M(\mathbb{R})$.

• The branching mechanism $\phi$:

$$\phi(\lambda) = b\lambda + c\lambda^2/2 + \int_{0}^{\infty} (e^{-z\lambda} - 1 + z\lambda)m(dz).$$

• $M(\mathbb{R})$-valued $\{ X_t \}$ process is called a super-Lévy process if

\[
\left\{ \begin{array}{l}
\mathbb{E}_{\mu}\left\{ \exp\left[ -\langle X_t, f \rangle \right] \right\} = \exp\left\{ -\langle \mu, v_t \rangle \right\}, \\
\frac{\partial}{\partial t} v_t(x) = Av_t(x) + \phi(v_t(x)), \quad v_0(x) = f(x).
\end{array} \right.
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• $D(\mathbb{R}) := \{f \in C^0 \cap L^\infty : f \text{ is bounded right continuous increasing and } f(-\infty) = 0\}$. 

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\end{array} \right.$$
Our aim in this talk is that under a mild condition on $A$, \{${Y_t}$\}, defined by $Y_t(x) = X_t(-\infty, x]$, is the pathwise unique solution to

$$Y_t(x) = Y_0(x) - b \int_0^t Y_s(x) ds + \sqrt{c} \int_0^t \int_0^{Y_s(x)} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Y_s-(x)} z \tilde{N}_0(ds, dz, du) + \int_0^t A^* Y_s(x) ds,$$  \hspace{1cm} (2)

where $W(ds, du)$ is a GWN and $\tilde{N}_0(ds, dz, du)$ compensated Poisson random measure (CPRM), $A^*$ denotes the dual operator of $A$.

- Xiong (2012): $A = \Delta/2$ and $b = \tilde{N}_0 = 0$.
- Key approach: connecting (2) with a backward doubly SDE. Xiong (2012) used an $L^2$-argument. We use an $L^1$-argument.
- For $M(\mathbb{R})$-valued process \{${X_t}$\}, its distribution \{${Y_t}$\} is $D(\mathbb{R})$-valued.
Our aim in this talk is that under a mild condition on $A$, $\{Y_t\}$, defined by $Y_t(x) = X_t(\infty, x]$, is the pathwise unique solution to

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- For \( M(\mathbb{R}) \)-valued process \( \{X_t\} \), its distribution \( \{Y_t\} \) is \( D(\mathbb{R}) \)-valued.
Main results

**Theorem 1**

$D(\mathbb{R})$-valued process $\{Y_t\}$ is the distribution of a super-Lévy process iff there is, on an enlarged probability space, a GWN $\{W(ds, du)\}$ and a CPRM $\{\tilde{N}_0(ds, dz, du)\}$ so that $\{Y_t\}$ solves (2).

Let $(P_t)_{t \geq 0}$ be the transition semigroup of a Lévy process with generator $A$.

**Condition 1**

For some continuous function $(t, z) \mapsto p_t(z)$, $\alpha \in (0, 1)$ and $C \in B[0, \infty)$,

$$P_t(x, dy) = p_t(y - x)dy \quad \text{and} \quad p_t(x) \leq t^{-\alpha}C(t), \quad t > 0, \ x, y \in \mathbb{R}.$$  

The condition holds if $A$ is the generator of a stable process with index in $(1, 2]$.

**Theorem 2**

Under Condition 1, the pathwise uniqueness holds for (2) with $Y_0 \in D(\mathbb{R})$. 
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**Theorem 2**

Under Condition 1, the pathwise uniqueness holds for (2) with $Y_0 \in D(\mathbb{R})$. 

Proof of Theorem 2

• Define $\xi$ by

$$\xi(t) = \beta t + \sigma B_t + \int_0^t \int_{\{|z| \leq 1\}} z \tilde{M}(ds, dz) + \int_0^t \int_{\{|z| > 1\}} z M(ds, dz)$$

(3)

and independent of $\{W(ds, du)\}$ and $\{\tilde{N}_0(ds, dz, du)\}$ and $\xi_t^r = \xi(r \wedge t) - \xi(t)$.

• Take $T > 0$ and define GWN $W^T(ds, dx)$ and CPRM $\tilde{N}_0^T(ds, dz, du)$ by

$$W^T((0, t] \times A) = W([T - t, T) \times A), \quad \tilde{N}_0^T((0, t] \times B) = \tilde{N}_0([T - t, T) \times B).$$

From (2),

$$Y_{T-t}(x) = Y_0(x) + \int_t^T A^* Y_{T-s}(x) ds + \sqrt{c} \int_{t-}^T \int_0^{Y_{T-s}(x)} W_T(ds, du)$$

$$- \int_t^T b Y_{T-s}(x) ds + \int_{t-}^T \int_0^{Y_{(T-s)^-}(x)} \tilde{N}_T(ds, dz, du).$$

(4)

$W_T(ds, du)$ is the backward Itô’s integral, i.e., in the Riemann sum approximating the stochastic integral, taking right end-points instead of the left ones.
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$W_T(ds, du)$ is the backward Itô’s integral, i.e., in the Riemann sum approximating the stochastic integral, taking right end-points instead of the left ones.
From (3) and (4), under Condition 1 for all $x \in \mathbb{R}$ and $0 \leq r \leq t \leq T$ we have a.s.

$$
Y_{T-t}(\xi^r_t + x) = Y_0(\xi^r_T + x) - b \int_t^T Y_{T-s}(\xi^r_s + x) ds + \sigma \int_t^T \nabla Y_{T-s}(\xi^r_s + x) dB_s
$$

$$
+ \sqrt{c} \int_{t-}^T \int_0^\infty W_T(ds, du)
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+ \int_{t-}^T \int_0^\infty \int_{t-}^\infty \int_0^\infty Y_{(T-s)_-}(\xi^r_s + x) z \tilde{N}_T(ds, dz, du)
$$

$$
- \int_t^T \int_{\mathbb{R}^o} [Y_{T-s}(\xi^r_s_- + x - z) - Y_{T-s}(\xi^r_s_- + x)] \tilde{M}(ds, dz). \quad (5)
$$

Remark:

(i) The fourth and fifth terms are time-reversed martingales.

(ii) We cannot establish (5) simultaneously for all $(t, x) \in [r, T] \times \mathbb{R}$. $t \mapsto Y_{T-t}(\xi^r_s + x)$ is neither right continuous nor left continuous.

(iii) The process defined by above general kind of SDE is unique.

(iv) Prove a generalized Itô’s formula, which is initiated by Pardoux and Peng (1994).
From (3) and (4), under Condition 1 for all $x \in \mathbb{R}$ and $0 \leq r \leq t \leq T$ we have a.s.

$$Y_{T-t}(\xi_t^r + x) = Y_0(\xi_T^r + x) - b \int_t^T Y_{T-s}(\xi_s^r + x) ds +\sigma \int_t^T \nabla Y_{T-s}(\xi_s^r + x) dB_s$$

$$\quad + \sqrt{c} \int_{t^-}^{T^-} \int_0^\infty W_T(ds, du)$$

$$\quad + \int_{t^-}^T \int_0^\infty \int_0^\infty Y_{(T-s)-}(\xi_s^r + x) z\tilde{N}_T(ds, dz, du)$$

$$\quad - \int_t^T \int_{\mathbb{R}^c} [Y_{T-s}(\xi_s^- + x - z) - Y_{T-s}(\xi_s^r + x)] \tilde{M}(ds, dz).$$ \hspace{1em} (5)

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$$+ \sqrt{c} \int_t^{T-} \int_0^{Y_{T-t}(\xi^r_s + x)} W_T(ds, du)$$

$$+ \int_t^{T-} \int_0^\infty \int_0^{Y(T-s)-}(\xi^r_s + x) z \tilde{N}_T(ds, dz, du)$$

$$- \int_t^T \int_{\mathbb{R}^\circ} [Y_{T-s}(\xi^r_{s-} + x - z) - Y_{T-s}(\xi^r_{s-} + x)] \tilde{M}(ds, dz). \quad (5)$$

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+ \sqrt{c} \int_t^T \int_0^{T-t} \left( Y_{T-t}(\xi_s^r + x) \right) W_T(ds, du) \\
+ \int_t^T \int_0^\infty \int_0^\infty \left( Y_{(T-s)}(\xi_s^r + x) \right) z\tilde{N}_T(ds, dz, du) \\
- \int_t^T \int_{\mathbb{R}^\circ} \left[ Y_{T-s}(\xi_s^- + x - z) - Y_{T-s}(\xi_s^- + x) \right] \tilde{M}(ds, dz).
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The weak solution for the following SPDE was constructed by Mytnik (2002):

\[
\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + X_t(x)^{\beta} \dot{L}, \quad X_0 \geq 0, \ x \in \mathbb{R}^d, \tag{6}
\]

where \( L(ds, dx) \) is a one-sided, \( \alpha \)-stable white noise without negative jumps, \( 1 < \alpha < \min(2, (2/d) + 1), \ \beta > 0, \ p := \alpha \beta < (2/d) + 1. \)

• \( p = 1 \), the solution is a superprocess and the weak uniqueness holds.

• \( p \neq 1 \), the uniqueness for (6) and the properties of solution are unknown.

• We consider the case \( d = 1 \) and \( p \in (0, \alpha) \) here. Other cases are being considered.
Further result: SPDE driven by $\alpha$-stable noise

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where $L(ds, dx)$ is a one-sided, $\alpha$-stable white noise without negative jumps, $1 < \alpha < \min(2, (2/d) + 1)$, $\beta > 0$, $p := \alpha \beta < (2/d) + 1$.

- $p = 1$, the solution is a superprocess and the weak uniqueness holds.
- $p \neq 1$, the uniqueness for (6) and the properties of solution are unknown.
- We consider the case $d = 1$ and $p \in (0, \alpha)$ here. Other cases are being considered.
Further result: SPDE driven by $\alpha$-stable noise

The weak solution for the following SPDE was constructed by Mytnik (2002):

$$\frac{\partial X_t(x)}{\partial t} = \frac{1}{2} \Delta X_t(x) + X_t(x)\beta \dot{L}, \quad X_0 \geq 0, \ x \in \mathbb{R}^d, \quad (6)$$

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• Equation (6) means:
\[
\langle X_t, f \rangle = \langle X_0, f \rangle + \frac{1}{2} \int_0^t \langle X_s, f'' \rangle ds + \int_0^t \int_{\mathbb{R}} X_{s-}(x)^{\beta} f(x) L(ds, dx).
\] (7)

• \( \{X_t\} \) satisfies SPDE (7) iff it satisfies
\[
\langle X_t, f \rangle = \langle X_0, f \rangle + \frac{1}{2} \int_0^t \langle X_s, f'' \rangle ds + \int_0^t \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} X_{s-}(u)^p z f(u) \tilde{N}_0(ds, dz, du, dv),
\] (8)
where \( \tilde{N}_0(ds, dz, du, dv) \) is a CPRM.

• Similar to Theorem 1.1 (a) and 1.3 (a) in Mytnik and Perkins (2003) we have: \( X_t(\cdot) \) has a continuous version for fixed \( t \).
Occupation density \( \mathcal{V}_t(x) := \int_0^t X_s(x) ds \) has a jointly continuous version.

• Connecting (8) with a backward doubly SDE, (8) has a pathwise uniqueness solution, which implies the weak uniqueness to (7).
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Thanks!

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