Branes, Quivers and BPS algebras

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3.10. Recapitulation

- We have argued that the derived category of coherent sheaves form a good model of branes and their bound states with morphisms encoding the spectrum of open strings.
- Studying morphisms in our category, we derived a class of supersymmetric quantum mechanics labelled by a framed quiver with potential.
- We have seen that after a deformation by Ω-background, the supersymmetric quantum mechanics describing the low energy behavior of n D0-branes bound to a D2-, D4- and D6-brane has vacua labelled by $1d$, $2d$ and $3d$ partitions respectively.
- Today, we are going to introduce an algebraic describing processes of bounding/separating D0-branes from a give bound-state. More concretely, we are going to see how to use the correspondence $M(n+1,n)$ from yesterday to construct a module structure on the space of our BPS vacua.

4. Modules from correspondences

4.1. Rising and lowering generators

- The goal is to define a geometric action of a two copies of the cohomological Hall algebra (rising and lowering generators) increasing and decreasing the number of D0-branes. [Nakajima (1984), Kontsevich-Soibelman (2010),. . .]
- In particular, we are now going to define

$$e_m : H^*_{U(1)^2}(M(n), \mathrm{Crit}(W)) \to H^*_{U(1)^2}(M(n+1), \mathrm{Crit}(W))$$
$$f_m : H^*_{U(1)^2}(M(n+1), \mathrm{Crit}(W)) \to H^*_{U(1)^2}(M(n), \mathrm{Crit}(W))$$

where $M(n) = \mathcal{M}(n)/GL(n)$ and $\mathcal{M}(n)$ is the space of stable quiver representations

$$(B_1, B_2, B_3, I) \qquad (B_1, B_2, B_3, I, J) \qquad (B_1, B_2, B_3, I, J_1, J_2)$$

for the framed moduli space associated to D2, D4 and D6 respectively and the circular node of dimension n.

4.2. Affine Yangian of $\mathfrak{gl}_1$

- Introducing generators

$$\psi_{m+n} = [e_m, f_n]$$

  the triple e_n, f_n, ψ_n can be shown to satisfy relations of (shifted) $\mathfrak{gl}_1$ affine Yangian (see e.g. [Tsymbaliuk (2014)]) and different choices of framings lead to its different representations [MR-Soibelman-Yang-Zhao (2020)].
- In this section, we are going to construct such representations for the three elementary framings.
- If we started from a different geometry than $\mathbb{C}^3$, we would likely discover the Quiver Yangians from [Li-Yamazaki (2020)] and their representations [Galakhov-Li-Yamazaki (2021)].

4.4. Nakajima's rising and lowering operators

- At the end of the last lecture, we defined a correspondence

$$\begin{array}{ccc} & M(n+1,n) & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ M(n+1) & & M(n) \end{array}$$

- Starting with $\alpha \in H^*_{U(1)^2}(M(n), \mathrm{Crit}(W))$, we can now define

$$e_0\alpha = p_*(q^*(\alpha))$$

  by pulling it back by q and pushing forward by q and obtain an element in $H^*_{U(1)^2}(M(n+1), \mathrm{Crit}(W))$.
- Reversing the order of the two maps, we have

$$f_0\alpha \quad = \quad q_*(p^*(\alpha))$$

- Utilizing the tautological line bundle, we can define

$$\begin{aligned} e_m\alpha &= p_*(c_1(L^m) \wedge q^*(\alpha)) \\ f_m\alpha &= q_*(c_1(L^m) \wedge p^*(\alpha)) \end{aligned}$$

4.7. Fixed-points basis

- Remember that we have an isomorphism of equivariant cohomologies of the form

$$
\begin{aligned}
\oplus_{\lambda\in F_n}\mathbb{C}[\epsilon_1,\epsilon_2]|\lambda\rangle &\rightarrow H^*_{U(1)^2}(M(n),\mathrm{Crit}(W)) \\
\oplus_{(\lambda,\lambda+\square)\in F_{n+1,n}}\mathbb{C}[\epsilon_1,\epsilon_2]|\lambda,\lambda+\square\rangle &\rightarrow H^*_{U(1)^2}(M(n+1,n),\mathrm{Crit}(W))
\end{aligned}
$$

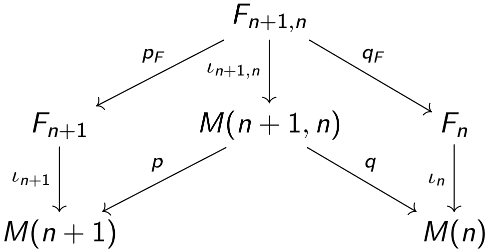
  with F_n being the fixed-point set of $M(n)$ and $F_{n+1,n}$ the fixed-point set of $M(n+1,n)$.
- We also have the embeddings of fixed points

$$
\begin{aligned}
\iota_\lambda &: \lambda \hookrightarrow M(n) \text{ for } \lambda\in F_n \\
\iota_{\lambda,\lambda+\square} &: \lambda \hookrightarrow M(n+1,n) \text{ for } \lambda\in F_{n+1,n}
\end{aligned}
$$

- Pushing forward generators $|\lambda\rangle \in H^*_{U(1)^2}(\lambda)$ and $|\lambda,\lambda+\square\rangle \in H^*_{U(1)^2}((\lambda,\lambda+\square))$ by these maps thus produces a natural fixed-point basis of $H^*_{U(1)^2}(M(n),\mathrm{Crit}(W))$ and $H^*_{U(1)^2}(M(n+1,n),\mathrm{Crit}(W))$ respectively.

4.8. Action in fixed-points basis

- We can now consider the following diagram



- Using the above correspondence, we can now find the action of e_n, f_n in the fixed-point basis given by $|\lambda\rangle \in H^*_{U(1)^2}(\lambda)$ as

$$
\begin{aligned}
e_m|\lambda\rangle &= \iota^{-1}_{n+1*} \circ p_* \circ c_1(L^m) \wedge q^* \circ \iota_{n*}|\lambda\rangle \\
f_m|\lambda+\Box\rangle &= \iota^{-1}_{n*} \circ q_* \circ c_1(L^m) \wedge p^* \circ \iota_{n+1*}|\lambda+\Box\rangle
\end{aligned}
$$

4.9. Atiyah-Bott localization formula

- Let λ be a fixed point in $M(n)$, then we can invert the push-forward of the embedding $\iota_\lambda : \lambda \hookrightarrow M(n)$ as

$$\iota_{\lambda *}^{-1} = \frac{\iota_\lambda^*}{e_{U(1)^2}(T_\lambda^* M(n))}$$

where $T_\lambda^* M(n)$ is the tangent space of $M(n)$ at λ and $e_{U(1)^2}(\cdot)$ encodes its weights as a $U(1)^2$ representation.

- More concretely, $T_\lambda^* M(n)$ splits into the direct sum of $U(1)^2$ representations

$$T_\lambda^* M(n) = \oplus_{\alpha=1}^{\dim\ M(n)} \mathbb{C}_{\epsilon_\alpha}$$

where $U(1)^2$ acts on $\mathbb{C}_{\epsilon_\alpha}$ as

$$z \to e^{i\epsilon_\alpha} z$$

- The character is then given by the product

$$e_{U(1)^2}(T_\lambda(\mathcal{M}))) = \prod_{i=1}^{\dim\ M(n)} \epsilon_\alpha$$

4.10. Towards explicit formula

- Let us now sketch the calculation for $e_0|\lambda\rangle$.
- Commuting the push-forward maps

$$\iota_{n+1*}^{-1} \circ p_* \circ q^* \circ \iota_{n*}|\lambda\rangle \quad = \quad p_{F*} \circ \iota_{n+1,n*}^{-1} \circ q^* \circ \iota_{n*}|\lambda\rangle$$

- The Atiyah-Bott localization formula leads to

$$= \sum_{(\mu+\Box,\mu)\in F_{n+1,n}} \frac{p_{F*} \circ i_{\mu+\Box,\mu}^* \circ q^* \circ \iota_{\lambda*}}{e_{U(1)^3}(T_{\mu+\Box,\mu}^* M(n+1,n))}|\lambda\rangle$$

- Commuting the pull-back maps gives

$$= \sum_{(\mu+\Box,\mu)\in F_{n+1,n}} \frac{p_{F*} \circ q_{\mu+\Box,\mu}^* \circ \iota_{\mu}^* \circ \iota_{\lambda*}}{e_{U(1)^3}(T_{\mu+\Box,\mu}^* M(n+1,n))}|\lambda\rangle$$

- Using the Atiyah-Bott localization formula again

$$\sum_{(\mu+\Box,\mu)\in F_{n+1,n}} \frac{e_{U(1)^3}(T_{\mu}^* M(n))}{e_{U(1)^3}(T_{\mu+\Box,\mu}^* M(n+1,n))} p_{F*} \circ q_{\mu+\Box,\mu}^* \circ \iota_{\mu*}^{-1} \circ \iota_{\lambda*}|\lambda\rangle$$

- The composition of $\iota_{\mu*}^{-1}$ with $\iota_{\lambda*}$ vanishes unless $\lambda = \mu$ and

$$= \sum_{(\lambda+\square,\lambda)\in F_{n+1,n}} \frac{e_{U(1)^2}(T^*_\lambda M(n))}{e_{U(1)^2}(T^*_{\lambda+\square,\lambda} M(n+1,n))} p_{F*} \circ q^*_{\lambda+\square,\lambda} |\lambda\rangle$$

leading to the final expression

$$= \sum_{(\lambda+\square,\lambda)\in F_{n+1,n}} \frac{e_{U(1)^2}(T^*_\lambda M(n))}{e_{U(1)^2}(T^*_{\lambda+\square,\lambda} M(n+1,n))} |\lambda+\square\rangle$$

- Analogously for other operators, the final formulas read

$$e_m|\lambda\rangle = \sum_{(\lambda,\lambda+\square)\in F_{n+1,n}} \frac{\epsilon^m_\square e_{U(1)^2}(T^*_\lambda M(n))}{e_{U(1)^2}(T^*_{\lambda+\square,\lambda} M(n+1,n))} |\lambda+\square\rangle$$

$$f_m|\lambda+\square\rangle = \sum_{(\lambda,\lambda+\square)\in F_{n+1,n}} \frac{\epsilon^m_\square e_{U(1)^2}(T^*_{\lambda+\square} M(n+1))}{e_{U(1)^2}(T^*_{\lambda+\square,\lambda} M(n+1,n))} |\lambda\rangle$$

4.11. Character of the tangent space

- At a given fixed point, the vector space $\mathbb{C}^n$ (and its dual) decomposes into weights according to the Young diagram

$$V_\lambda = \oplus_{\square\in\lambda}\mathbb{C}_{\epsilon_\square} \qquad V_\lambda^* = \oplus_{\square\in\lambda}\mathbb{C}_{-\epsilon_\square}$$

- The tangent space at a point λ is then given by the cohomology of the (equivariant) tangent complex

$$V_\lambda^*\otimes V_\lambda \xrightarrow{([B_i,\xi],\xi I,\xi^{-1}J)} V_\lambda^*\otimes V_\lambda\otimes(\mathbb{C}_{\epsilon_1}^*+\mathbb{C}_{\epsilon_2}^*+\mathbb{C}_{\epsilon_3}^*)+V_\lambda+V_\lambda\otimes\mathbb{C}_\epsilon^*$$

- As a $U(1)^2$ representation, it decomposes as

$$T_\lambda = (\mathbb{C}_{\epsilon_1}^*+\mathbb{C}_{\epsilon_2}^*+\mathbb{C}_{\epsilon_3}^*-1)\otimes V_\lambda\otimes V_\lambda^*+V_\lambda+V_\lambda\otimes\mathbb{C}_\epsilon^*$$

- Using $\mathbb{C}_\epsilon\otimes\mathbb{C}_{\tilde\epsilon}=\mathbb{C}_{\epsilon+\tilde\epsilon}$ we can simplify the above expression into a sum of $\mathbb{C}_{\epsilon_\alpha}$ terms and write the character as the product of $U(1)^2$ weights

$$\prod_\alpha \epsilon_\alpha$$

- Computation of the tangent space to the correspondence at $(\lambda + \Box, \lambda)$ is slightly more complicated (see e.g. [Nakajima (1984)]). It decomposes as a $U(1)^2$ representation as

$$T_\lambda + T_{\lambda+\Box} - N_{\lambda+\Box,\lambda}$$

where

$$N_{\lambda+\Box,\lambda} = (\mathbb{C}^*_{\epsilon_1} + \mathbb{C}^*_{\epsilon_2} + \mathbb{C}^*_{\epsilon_3} - 1) \otimes V_\lambda \otimes V^*_{\lambda+\Box} + V_\lambda + V^*_{\lambda+\Box} \oplus \mathbb{C}^*_\epsilon$$

is the decomposition into the $U(1)^2$ representation of the normal bundle of $M(n+1,n)$ inside $M(n+1) \times M(n)$ at $(\lambda + \Box, \lambda)$.

- As a result, we can write

$$\begin{aligned} e_m|\lambda\rangle &= \sum_{(\lambda,\lambda+\Box)\in F_{n+1,n}} \epsilon_\Box^m e_{U(1)^2}(N_{\lambda+\Box,\lambda} - T_{\lambda+\Box})|\lambda+\Box\rangle \\ f_m|\lambda+\Box\rangle &= \sum_{(\lambda,\lambda+\Box)\in F_{n+1,n}} \epsilon_\Box^m e_{U(1)^2}(N_{\lambda+\Box,\lambda} - T_\lambda)|\lambda\rangle \end{aligned}$$

4.12. D2-brane and the vector representation

- Let us now explicitly evaluate the above expressions in the case of the D2-moduli space.
- According to the above formulas, $f_m|n+1\rangle$ is given by

$$(n\epsilon_1)^m \frac{e(V_n \otimes V_{n+1}^* \otimes (\mathbb{C}_{\epsilon_1}^* + \mathbb{C}_{\epsilon_2}^* + \mathbb{C}_{\epsilon_3}^* - 1) + V_n + (\mathbb{C}_{\epsilon_1+\epsilon_2}^* + \mathbb{C}_{\epsilon_1+\epsilon_3}^*) \otimes V_{n+1}^*)}{e(V_n \otimes V_n^* \otimes (\mathbb{C}_{\epsilon_1}^* + \mathbb{C}_{\epsilon_2}^* + \mathbb{C}_{\epsilon_3}^* - 1) + V_n + (\mathbb{C}_{\epsilon_1+\epsilon_2}^* + \mathbb{C}_{\epsilon_1+\epsilon_3}^*) V_n^*)} |n\rangle$$

$$(n\epsilon_1)^m (n\epsilon_1 - \epsilon_3)(n\epsilon_1 - \epsilon_2) \prod_{i=1}^{n} \frac{(n\epsilon_1 - (i-1)\epsilon_1 + \epsilon_1)(n\epsilon_1 - (i-1)\epsilon_1 + \epsilon_2)(n\epsilon_1 - (i-1)\epsilon_1 + \epsilon_3)}{n\epsilon_1 - (i-1)\epsilon_1} |n\rangle$$

- Analogously $e_m|n\rangle$ is given by

$$(n\epsilon_1)^m \frac{e(V_n \otimes V_{n+1}^* \otimes (\mathbb{C}_{\epsilon_1} + \mathbb{C}_{\epsilon_2} + \mathbb{C}_{\epsilon_3} - 1) + V_n + (\mathbb{C}_{\epsilon_1+\epsilon_2} + \mathbb{C}_{\epsilon_1+\epsilon_3}) \otimes V_{n+1}^*)}{e(V_{n+1} \otimes V_{n+1}^* \otimes (\mathbb{C}_{\epsilon_1}^* + \mathbb{C}_{\epsilon_2}^* + \mathbb{C}_{\epsilon_3}^* - 1) + V_{n+1} + (\mathbb{C}_{\epsilon_1+\epsilon_2}^* + \mathbb{C}_{\epsilon_1+\epsilon_3}^*) V_{n+1}^*)} |n+1\rangle$$

$$(n\epsilon_1)^{m-1} \prod_{i=1}^{n+1} \frac{n\epsilon_1 - (i-1)\epsilon_1}{(n\epsilon_1 - (i-1)\epsilon_1 - \epsilon_1)(n\epsilon_1 - (i-1)\epsilon_1 - \epsilon_2)(n\epsilon_1 - (i-1)\epsilon_1 - \epsilon_3)} |n+1\rangle$$

- We can now simplify the products as

$$f_m|n+1\rangle = (n+1)(n\epsilon_1)^m \prod_{i=1}^{n+1}((n+1-i)\epsilon_1-\epsilon_2)((n+1-i)\epsilon_1-\epsilon_3)|n\rangle$$

$$e_m|n\rangle = -\frac{1}{\epsilon_1}(n\epsilon_1)^m \prod_{i=1}^{n+1}\frac{1}{((n+1-i)\epsilon_1-\epsilon_2)((n+1-i)\epsilon_1-\epsilon_3)}|n+1\rangle$$

- Let us introduce

$$A_k = \prod_{i=1}^{k}\frac{1}{((k+1-i)\epsilon_1-\epsilon_2)((k+1-i)\epsilon_1-\epsilon_3)}$$

and renormalize

$$|\tilde{m}\rangle = \prod_{k=1}^{m} A_k|m\rangle$$

- In terms of the renormalized basis, we have

$$f_n|m+1\rangle = (m+1)(m\epsilon_1)^n|m\rangle$$

$$e_n|m\rangle = -\frac{1}{\epsilon_1}(m\epsilon_1)^n|m+1\rangle$$

- If we identify

$$|m\rangle = z^m$$

the above action factors through

$$\begin{aligned} f_n &\to (\epsilon_1 z\partial)^n \partial \\ e_n &\to \frac{1}{\epsilon_1} z(\epsilon_1 z\partial)^n \end{aligned}$$

acting on $\mathbb{C}[z]$.

- We have ended up with a geometric construction of so-called vector representation of the 1-shifted affine Yangian.

4.13. D4-brane and the Fock representation

- One can perform the same calculation for the D4-brane framing and obtain the Fock representation of the affine Yangian. Instead of doing the algebra let us simply state the result.
- Let us introduce an associative algebra

$$[J_m, J_n] = -\frac{1}{\epsilon_1 \epsilon_2} m \delta_{m,-n}$$

- This algebra is known as the $\widehat{\mathfrak{gl}}_1$ Kac-Moody algebra, the $\mathfrak{gl}_1$ current algebra or the Heisenberg vertex operator algebra.
- It turns out that this algebra can be given a structure of a vertex operator algebra. Generally, configurations of D4-branes are always expected to lead to a vertex operator algebra leading to an interesting interplay between the theory of VOAs and the geometry of divisors in Calabi-Yau threefolds. [Procházka-MR (2018)]

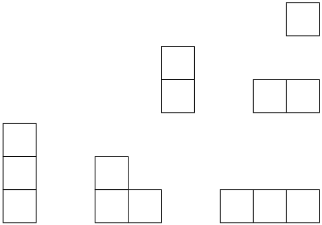
- $\widehat{\mathfrak{gl}}_1$ admits a class of highest-weight modules generated by the action of negative modes J_n on $|\mu\rangle$ satisfying

$$J_0|\mu\rangle = \mu|\mu\rangle, \qquad J_m|\mu\rangle = 0, \quad \text{for } m > 0$$

- The modules are generated by

$$\begin{array}{rrr} & & J_{-1}|\mu\rangle \\ & J_{-1}^2|\mu\rangle, & J_{-2}|\mu\rangle \\ J_{-1}^3|\mu\rangle, & J_{-1}J_{-2}|\mu\rangle, & J_{-3}|\mu\rangle \end{array}$$

- Note also that these are in correspondence with 2d partitions



- An alternative basis of the Yangian is formed by $[e_1, e_2]$ and

$$\tilde{J}_{-n} = \frac{1}{(m-1)!}\mathrm{ad}_{e_1}^{m-1} e_0, \qquad \tilde{J}_n = \frac{1}{(m-1)!}\mathrm{ad}_{f_1}^{m-1} f_0$$

- The geometric action constructed above factors through

$$Y_{0,0,1} \quad : \quad \tilde{J}_n \to J_n$$

$$Y_{0,0,1} \quad : \quad [e_1, e_2] \to \frac{\epsilon_1^2\epsilon_2^2}{3}\sum_{k,m=-\infty}^{\infty} : J_{-m-k-2}J_mJ_k : + \frac{\epsilon_1\epsilon_2\epsilon_3}{2}\sum_{m=1}^{\infty} mJ_{-m-1}J_{m-1}$$

  acting on the above Fock module for $\mu = 0$.
- It is also possible to recover the general μ by introducing an equivariant parameter associated to the $GL(1)$ action of the framing node. This refinement turns out to be essential for understanding representations associated to more complicated configurations of D4-branes as we are going to sketch in the next section.

4.14. D6-brane and the MacMahon

- Analogously, one can construct a representation of a -1-shifted affine Yangian on a vector space labeled by 3d partitions associated to the D6-brane framing. It is straightforward to find explicit relations and recover the formulas from [MR-Soibelman-Yang-Zhao (2020)]. Restricting to the non-shifted Yangian, this module is conjecturally equivalent to the module constructed algebraically in [Feigin-Jimbo-Miwa-Mukhin (2012), Procházka (2015)].

5. Cherednik and $\mathcal{W}$ algebras

5.1. Affine Yangian

- e_m, f_m, ψ_m for D4-branes satisfy relations of the $\mathfrak{gl}_1$ affine Yangian [Schiffmann-Vasserot (2012), Maulik-Okounkov (2012)]:

$$
\begin{aligned}
\psi_{i+j} &= [e_i, f_j] \qquad [\psi_i, \psi_j] = 0 \\
0 &= [e_{i+3}, e_j] - 3[e_{i+2}, e_{j+1}] + 3[e_{i+1}, e_{j+2}] - [e_i, e_{j+3}] \\
&\quad + \sigma_2[e_{i+1}, e_j] - \sigma_2[e_i, e_{j+1}] - \sigma_3\{e_i, e_j\} \\
0 &= [f_{i+3}, f_j] - 3[f_{i+2}, f_{j+1}] + 3[f_{i+1}, f_{j+2}] - [f_i, f_{j+3}] \\
&\quad + \sigma_2[f_{i+1}, f_j] - \sigma_2[f_i, f_{j+1}] + \sigma_3\{f_i, f_j\} \\
0 &= [\psi_{i+3}, e_j] - 3[\psi_{i+2}, e_{j+1}] + 3[\psi_{i+1}, e_{j+2}] - [\psi_i, e_{j+3}] \\
&\quad + \sigma_2[\psi_{i+1}, e_j] - \sigma_2[\psi_i, e_{j+1}] - \sigma_3\{\psi_i, e_j\} \\
0 &= [\psi_{i+3}, f_j] - 3[\psi_{i+2}, f_{j+1}] + 3[\psi_{i+1}, f_{j+2}] - [\psi_i, f_{j+3}] \\
&\quad + \sigma_2[\psi_{i+1}, f_j] - \sigma_2[\psi_i, f_{j+1}] + \sigma_3\{\psi_i, f_j\}
\end{aligned}
$$

$$
[\psi_0, e_i] = [\psi_0, f_i] = [\psi_1, e_i] = [\psi_1, f_i] = 0
$$

$$
[\psi_2, e_i] = 2e_i \qquad [\psi_2, f_i] = -2f_i
$$

$$
\mathrm{Sym}_{i,j,k}[e_i, [e_j, e_k]] = 0 \qquad \mathrm{Sym}_{i,j,k}[f_i, [f_j, f_k]] = 0
$$

- Its subalgebras generated by e_m restricted to $m \geq k$ together with all f_m, ψ_m are called k-shifted affine Yangians.
- With a little bit of work, one can also introduce shifted Yangians with negative shift $k < 0$ but let us not go into details. [MR-Soibelman-Yang-Zhao (2020)]
- More complicated representations of the $\mathfrak{gl}_1$ affine Yangian can be obtain by utilizing the coproduct

$$\Delta : \mathcal{Y} \to \mathcal{Y} \otimes \mathcal{Y}$$

given by formulas

$$\Delta \quad : \quad J_n \to J_n \otimes \mathbb{1} + \mathbb{1} \otimes J_n$$

$$\Delta \quad : \quad [e_2, e_1] \to [e_2, e_1] \otimes \mathbb{1} + \mathbb{1} \otimes [e_2, e_1] + \epsilon_1 \epsilon_2 \epsilon_3 \sum_{m=1}^{\infty} m J_{-m-1} \otimes J_{m-1}$$

[Schiffmann-Vasserot (2012), Maulik-Okounkov (2012)]

5.2. Corner vertex operator algebras

- Let us first compose the coproduct with the two Fock representations $Y_{0,0,1}$ acting on the Fock spaces $\mathcal{F}_{\mu_1} \otimes \mathcal{F}_{\mu_2}$:

$$(Y_{0,0,1} \otimes Y_{0,0,1}) \circ \Delta(t)$$

- The states of $\mathcal{F}_{\mu_1} \otimes \mathcal{F}_{\mu_2}$ are in correspondence with a pair of partitions that are in turn in correspondence with fixed points of the quiver moduli with rank-two framing if we introduce equivariant parameters μ_1, μ_2 associated to the Cartan of $GL(2)$ acting on the framing node.
- This is exactly the representation one gets from correspondences! [Schiffmann-Vasserot (2012), Maulik-Okounkov (2012), Yang-Zhao (2016)]

- One can also show that the above map produces only a subalgebra of the tensor product of $\widehat{\mathfrak{gl}}_1$ Kac-Moody algebras know as the Virasoro algebra tensored with a singe copy of the $\widehat{\mathfrak{gl}}_1$ Kac-Moody algebra generated by L_m, J_n such that

$$\begin{aligned} [J_m, J_n] &= -\frac{2}{\epsilon_1 \epsilon_2} \delta_{m,-n} \\ [L_m, J_n] &= -n J_{m+n} \\ [L_m, L_n] &= (m-n) L_{m+n} + \frac{1}{6} \left(7 + 3 \left(\frac{\epsilon_1}{\epsilon_2} + \frac{\epsilon_2}{\epsilon_1} \right) \right) n \left(n^2 - 1 \right) \delta_{m,-n} \end{aligned}$$

- The highest weight state then satisfies

$$J_m |\mu_1, \mu_2\rangle = L_m |\mu_1, \mu_2\rangle = 0, \quad \text{for } m > 0$$

and is an eigenstate of J_0, L_0 with eigenvalues depending on μ_1, μ_2.

- For special values of μ_1, μ_2, the above-constructed module is not irreducible. For example, specializing μ_1, μ_2 such that $J_0|\mu_1, \mu_2\rangle = L_0|\mu_1, \mu_2\rangle = 0$, we can define an irreducible module by further imposing

$$L_{-1}|\mu_1, \mu_2\rangle = 0$$

- This module has a geometric construction coming from turning on the Higgs vev on D4-branes! [Chuang-Creutzig-Diaconescu-Soibelman (2019)]

- One can proceed with a construction of more complicated algebras associated to a generic configuration of D4-branes by using the coproduct $N_1 + N_2 + N_3 - 1$ times and then composing with

$$Y_{1,0,0}^{\oplus N_1} \otimes Y_{0,1,0}^{\oplus N_1} \otimes Y_{0,0,1}^{\oplus N_3}$$

  leading to a class of corner vertex operator algebras [Gaiotto-MR (2017)] acting on the tensor product of $N_1 + N_2 + N_3$ Fock modules [Bershtein-Feigin-Merzon (2015), Litvinov-Spodyneiko (2016), Prochazka-MR (2018)].
- These modules have a geometric construction coming from intersecting D4-branes! [MR-Soibelman-Yang-Zhao (2018)]
- Turning on nilpotent Higgs vevs in general setting produces "pit" representations [Bershtein-Feigin-Merzon (2015), Gaiotto-MR (2017), Prochazka-MR (2017)] as shown in [Butson-MR (in progress)].

5.3. Cherednik algebras

- Let us finish with an exploration of a much-less-understood construction associated to more general configurations of D2-branes.
- Recall the elementary M2-brane representation $A_{1,0,0}$:

$$f_0 = \partial, \quad f_1 = \epsilon_1 z \partial^2, \quad f_2 = (\epsilon_1 z \partial)^2 \partial \quad e_0 = \frac{1}{\epsilon_1} z, \quad e_1 = z^2 \partial$$

- In the new basis, we have

$$J_n \to \frac{1}{\epsilon_1} z^n \qquad [f_0, f_1] \to \epsilon_1 \partial^2$$

- We can now use the coproduct and compose it with $A_{1,0,0} \otimes A_{1,0,0}$ to obtain

$$
\begin{aligned}
J_n &\rightarrow \frac{1}{\epsilon_1}(z_1^n + z_2^n) \\
[f_0, f_1] &\rightarrow \epsilon_1 \partial_1^2 + \epsilon_1 \partial_2^2 + \epsilon_1 \epsilon_2 \epsilon_3 \sum_{m=1}^{\infty} m \frac{z^{-m-1}}{\epsilon_1} \frac{z_2^{m-1}}{\epsilon_1} \\
&\rightarrow \epsilon_1 \partial_1^2 + \epsilon_1 \partial_2^2 + \frac{\epsilon_2 \epsilon_3}{\epsilon_1} \frac{2}{(z_1 - z_2)^2}
\end{aligned}
$$

- These expressions are known as a Dunkel representation of the Cherednik algebra associated to $\mathfrak{gl}(2)$. See e.g. the lecture notes [Opdam (2000)].

- Similarly, one can use the coproduct $N_1 + N_2 + N_3 - 1$ times and compose the result with

$$A_{1,0,0}^{\oplus N_1} \otimes A_{0,1,0}^{\oplus N_2} \otimes A_{0,0,1}^{\oplus N_3}$$

and obtain

$$\begin{aligned}
t_{0,d} &= \epsilon_1^{-1} \sum_{i=1}^{n_1} z_i^d + \epsilon_2^{-1} \sum_{i=1}^{n_2} (z_i')^d + \epsilon_3^{-1} \sum_{i=1}^{n_3} (z_i'')^d \\
t_{2,0} &= \epsilon_1 \sum_{i=1}^{n_1} \partial_{z_i}^2 + \frac{\epsilon_2 \epsilon_3}{\epsilon_1} \sum_{i<j} \frac{2}{(z_i - z_j)^2} + \epsilon_1 \sum_{i,j} \frac{2}{(z_i' - z_j'')^2} + \\
&\quad + \epsilon_2 \sum_{i=1}^{n_2} \partial_{z_i'}^2 + \frac{\epsilon_1 \epsilon_3}{\epsilon_2} \sum_{i<j} \frac{2}{(z_i' - z_j')^2} + \epsilon_2 \sum_{i,j} \frac{2}{(z_i - z_j'')^2} \\
&\quad + \epsilon_3 \sum_{i=1}^{n_3} \partial_{z_i''}^2 + \frac{\epsilon_1 \epsilon_2}{\epsilon_3} \sum_{i<j} \frac{2}{(z_i'' - z_j'')^2} + \epsilon_3 \sum_{i,j} \frac{2}{(z_i - z_j')^2}
\end{aligned}$$

that is a three-parametric generalization of the Cherednik algebra (and Calogero-Moser system) [MR-Gaiotto (2020)].