Remarks on Landau-Siegel Zeros

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Comparative Prime Number Theory Symposium

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Motivation and Background

- The Prime Number Theorem states that

$$\sum_{n \le x} \Lambda(n) \sim x, \quad \text{as} \quad x \to \infty,$$

where $\Lambda(n)$ denotes the Von-Mangoldt function.

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- Explicit formula :

$$\sum_{n \leq x} \Lambda(n) = x - \sum_{\rho:\zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

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- The Prime Number theorem follows from the fact that $\zeta(1 + it) \neq 0,\ t \in \mathbb{R}$.

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- Dirichlet (1837) introduced characters $\chi(\bmod D)$ to prove there are infinitely many primes $p \equiv a \pmod D, (a, D) = 1$. One key step in the proof is to show that $L(1, \chi) \neq 0$ for each non-principal character $\chi(\bmod D)$.

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- Fairly easy to show that $L(1, \chi) \neq 0$ if χ is complex (so that $\bar{\chi} \neq \chi$), but the non-vanishing of $L(1, \chi)$ for real characters χ is more subtle.

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- Fairly easy to show that $L(1, \chi) \neq 0$ if χ is complex (so that $\bar{\chi} \neq \chi$), but the non-vanishing of $L(1, \chi)$ for real characters χ is more subtle.
- To this end, Dirichlet developed his class number formula

$$L(1, \chi_D) = \frac{\pi h(-D)}{\sqrt{D}}, \quad D > 4.$$

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with effective constant.

- For some applications, lower bound $L(1, \chi) \gg D^{-1/2}$ is not strong enough.

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- Unconditionally, can show $L(1,\chi) \ll \log D$, but lower bounds are more difficult to obtain.

- Not able to rule out a real zero β of $L(s,\chi)$ with β close to $s=1$. Such a real zero β is a Landau-Siegel zero.

Motivation and Background

- Classical zero-free region shows $L(s,\chi)$ has at most one real zero β in region

$$\mathrm{Re}(s) \geq 1 - \frac{c}{\log(q(3+|\mathrm{Im}(s)|))}.$$

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- We say χ is an exceptional character, or that χ has a Landau-Siegel zero, if $L(\beta,\chi) = 0$ for some $\beta \geq 1 - c/\log q$.

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- We say χ is an exceptional character, or that χ has a Landau-Siegel zero, if $L(\beta, \chi) = 0$ for some $\beta \geq 1 - c/\log q$.
- We do not make constant $c > 0$ explicit, but it is fixed and effective.

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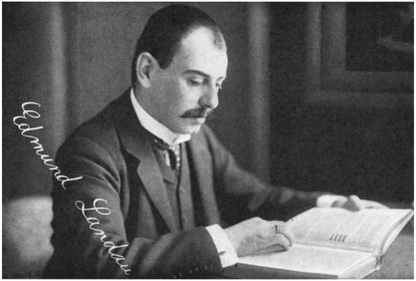
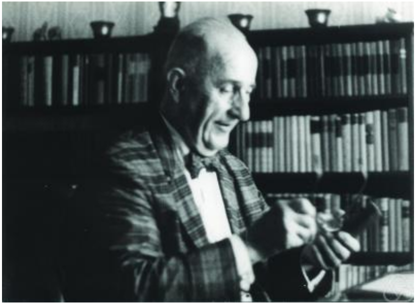
- Unfortunately, the proofs, in principle, doesn't allow for a determination of the constant in terms of ε.

Pictures



Edmund Landau



Carl Siegel

Pictures

Bemerkungen zum Heilbronnschen Satz

Edmund Landau

Acta Arithmetica 1 (1935) , 2-18

DOI: 10.4064/aa-1-1-2-18

Über die Classenzahl quadratischer Zahlkörper

Carl Siegel

Acta Arithmetica 1 (1935) , 83-86

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Refinements of Siegel's Theorem

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- Tatuzawa's (1952) refinement of Siegel's theorem says that for all $0 < \varepsilon < 1/2$, there is an effectively computable constant $q_0 = q_0(\varepsilon) > 0$ such that for $\chi \in \mathcal{S}$,
$$\beta \leq 1 - \frac{\varepsilon}{q^{\varepsilon}},$$
for all $q \geq q_0$ except at most one exceptional χ.

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- Further numerical refinements of Tatuzawa's result due to Hoffstein (1980), Ji-Lu (2004) and Chen (2007).

Refinements of Siegel's Theorem

- Sarnak and Zaharescu (2002) improved Tatuzawa's theorem assuming that if $\chi \in \mathcal{S}$, all non-real zeros of $L(s, \chi)$ lie on the 1/2-line.

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- Sarnak and Zaharescu (2002) improved Tatuzawa's theorem assuming that if $\chi \in \mathcal{S}$, all non-real zeros of $L(s, \chi)$ lie on the 1/2-line.

- Subject to this hypothesis, they showed that for any $\varepsilon > 0$, there exists an effectively computable constant $q_0 = q_0(\varepsilon) > 0$ such that for $\chi \in \mathcal{S}$,

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Refinements of Siegel's Theorem

- Sarnak and Zaharescu (2002) improved Tatuzawa's theorem assuming that if $\chi \in \mathcal{S}$, all non-real zeros of $L(s, \chi)$ lie on the $1/2$-line.
- Subject to this hypothesis, they showed that for any $\varepsilon > 0$, there exists an effectively computable constant $q_0 = q_0(\varepsilon) > 0$ such that for $\chi \in \mathcal{S}$,

$$\beta \leq 1 - \frac{1}{(\log q)^{\varepsilon}},$$

  for all $q \geq q_0$ except at most one exceptional χ.
- In particular, they "exponentiate" the quality of the zero free region at the cost of a hypothesis that, while assuming the generalized Riemann hypothesis for the non-real zeros, still permits the existence of Landau-Siegel zeros.

New Results

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Hypothesis (H_δ)

If $\chi \in \mathcal{S}$, then all the zeros of $L(s, \chi)$ in the disk $|z - 1| < \delta$ are real.

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Theorem (B.-Thorner-Zaharescu)

Fix $0 < \delta \leq 1/10$. Assume that H_δ is true. For any $\varepsilon > 0$, there exists an effectively computable constant $q_0 = q_0(\delta, \varepsilon) > 0$ such that for $\chi \in \mathcal{S}$,

$$\beta \leq 1 - \frac{1}{(\log q)^\varepsilon},$$

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- In contrast, the hypothesis assumed by Sarnak and Zaharescu has not been verified for any $\chi \in \mathcal{S}$.
- Our proof relies on a key refinement of Turán's power sum method.

Corollary (B.-Thorner-Zaharescu)

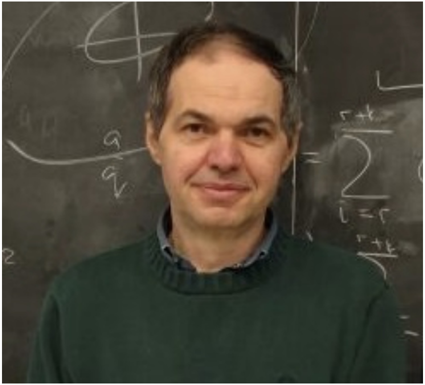
Fix $0 < \delta \leq 1/10$. Assume that H_δ is true. Then for all $\varepsilon > 0$, there exists an ineffective constant $c(\delta, \varepsilon) > 0$ such that if $\chi \in \mathcal{S}$ and

$$\beta > 1 - \frac{c(\delta, \varepsilon)}{(\log q)^{\varepsilon}},$$

then $L(\beta, \chi) \neq 0$.

Pictures

Jesse Thorner



Alexandru Zaharescu

Pictures

With Peter Sarnak

Thank you for your attention!