Spacing statistics of Farey Sequence

Bittu Chahal

Department of Mathematics
IIIT Delhi

(Based on joint work with Sneha Chaubey)

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Pair correlation of a sequence

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Let $\mathcal{F}$ be a finite set of cardinality N in $[0,1]$. The pair correlation measure $\mathcal{R}_{\mathcal{F}}(I)$ of a finite interval $I \subset \mathbb{R}$ is defined by

$$\frac{1}{N}\#\{(x,y) \in \mathcal{F}^2 : x \neq y,\ x-y \in \frac{1}{N}I + \mathbb{Z}\}.$$

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$$\mathcal{R}(I) = \lim_{n\to\infty} \mathcal{R}_{\mathcal{F}_n}(I).$$

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If

$$\mathcal{R}(I) = \int_I g(x)dx,$$

then g is called the limiting pair correlation function of $(\mathcal{F}_n)_n$.

Montgomery's pair correlation conjecture

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[Montgomery, 1973] conjectured that, for any fixed $\beta > 0$,

$$N(\beta, T) := \sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma-\gamma'\leq \frac{2\pi\beta}{\log T}}} 1 \sim \frac{T\log T}{2\pi}\int_0^\beta \left(1-\left(\frac{Sin\pi u}{\pi u}\right)^2\right) du,$$

as $T \to \infty$.

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[Montgomery, 1973] For $\alpha \in \mathbb{R}$ and $T \geq 2$ defined

$$F(\alpha) := F(\alpha, T) = \frac{2\pi}{T \log T} \sum_{0<\gamma,\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'),$$

where $w(u) = 4/(4 + u^2)$.

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He proved assuming RH that if $\alpha \in \mathbb{R}$ and $T \geq 2$ then $F(\alpha)$ is real, and $F(\alpha) = F(-\alpha)$. If $T > T_0(\epsilon)$ then $F(\alpha) \geq -\epsilon$ for all α. For fixed α satisfying $0 \leq \alpha < 1 - \epsilon$ we have

$$F(\alpha) = (1 + o(1))\,T^{-2\alpha}\log T + \alpha + o(1), \text{ as } T \to \infty.$$

[Montgomery, 1973] conjectured that for $\alpha \geq 1$,

$$F(\alpha) = 1 + o(1).$$

Farey Sequence

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Definition

Let Q be a positive integer and denote by $\mathcal{F}_Q$ the set of irreducible fractions in $[0, 1]$ whose denominator does not exceed Q,

$$\mathcal{F}_Q = \left\{ \frac{a}{q} : 0 \leq a \leq q \leq Q, (a, q) = 1 \right\}.$$

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Example

$$\mathcal{F}_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}.$$

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- The cardinality of $\mathcal{F}_Q$

$$N(Q) = 1 + \sum_{q=1}^{Q} \phi(q) = \frac{3Q^2}{\pi^2} + O\left(Q \log Q\right).$$

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- [Landau, 1924]

$$RH \iff \sum_{j=1}^{N(Q)} \delta^2(j) = O\left(Q^{-1+\epsilon}\right).$$

Pair correlation of Farey fractions

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Theorem (Boca and Zaharescu, 2005)

The pair correlation function of $(\mathcal{F}_Q)_Q$ *is given by*

$$g(\lambda) = \frac{6}{\pi^2\lambda^2} \sum_{1 \leq k < \frac{\pi^2\lambda}{3}} \phi(k) \log \frac{\pi^2\lambda}{3k}.$$

Moreover, as $\lambda \to \infty$

$$g(\lambda) = 1 + O\left(\lambda^{-1}\right).$$

Pair correlation of Farey fractions with divisibility constraints on denominators

- [Xiong and Zaharescu, 2008] studied the pair correlation of Farey fractions with prime denominators.
- [Xiong and Zaharescu, 2011] studied the pair correlation of Farey fractions with denominators coprime to B_Q.
- [Boca and Siskaki, 2022] studied the pair correlation of Farey fractions with denominators in some arithmetic progression.
- [.B and Chaubey, 2024] studied the pair correlation of Farey fractions with square-free denominators.

Visible lattice points along polynomials

Visible lattice points along polynomials

- For a fixed vector $\mathbf{c} = (c_n, c_{n-1}, \cdots, c_1) \in \mathbb{Z}^n$ with $c_n \neq 0, c_i \geq 0$ for all i, and $\gcd(c_n, c_{n-1}, \cdots, c_1) = 1$, let $P(x) = c_n x^n + \cdots c_1 x$, we define

$$V(\mathbf{c}) := \left\{ (a, b) \in \mathbb{N}^2 \;\middle|\; \begin{array}{l} b = qP(a) \text{ for some } q \in \mathbb{Q}^+, \ \nexists\ (a', b') \in \mathbb{N}^2 \\ \text{such that } b' = q'P(a'), \text{ and } a' < a, \ b' < b \end{array} \right\}$$

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- Denote
$$S = \{(a, b) \in \mathbb{N}^2 | \ \gcd(P(a), b) = 1\}.$$

- $S \subseteq V(\mathbf{c})$.

Polynomial Farey fractions

Let $\mathbf{c} = (c_1, \cdots, c_n) \in \mathbb{Z}^n$ be a fixed vector and $P(x) = c_n x^n + \cdots + c_1 x$. Define

$$\mathcal{F}_{Q,P} := \left\{ \frac{a}{q} \mid 1 \leq a \leq q \leq Q,\ \gcd(P(a), q) = 1 \right\}.$$

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If $P(x) = x(x+1)$ then for instance

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The cardinality of $\mathcal{F}_{Q,P}$

$$\mathcal{N}_{Q,P} = \#\mathcal{F}_{Q,P} = \frac{Q^2}{2} \prod_p \left(1 - \frac{f_P(p)}{p^2} \right) + O\left(Q^{1+\epsilon} \right),$$

where $f_P(p) = |\{1 \leq d \leq p |\ P(d) \equiv 0 \pmod{p}\}|$.

Pair correlation of Polynomial Farey fractions

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Theorem (.C, Chaubey, 2024)

Let $\mathbf{c} = (c_1, c_2) \in \mathbb{Z}^2_{>0}$ *be a fixed vector and* $P(x) = c_2 x^2 + c_1 x$. *The limiting pair correlation measure of the sequence* $(\mathcal{F}_{Q,P})_Q$ *under the GRH exists and is given by*

$$\mathcal{S}(\Lambda) \ll \frac{(c_1 c_2)^{\epsilon}}{\beta_P^{1+\epsilon}} \int_0^{\Lambda} \frac{1}{\lambda^{1-\epsilon}} \sum_{1 \le m < \frac{2\lambda}{\beta_P}} h_1(m) \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda,$$

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for any $\Lambda > 0$, *where* $\beta_P = \prod_p \left(1 - \frac{f_P(p)}{p^2}\right)$,
$f_P(p) = |\{1 \le d \le p : P(d) \equiv 0 \pmod p\}|$ *and*

$$h_1(m) = \frac{1}{m^{1+\epsilon}} \sum_{\substack{g_1|m \\ g_1|c_1}} \sum_{\substack{g_2|\frac{m}{g_1} \\ g_2|c_1}} \sigma\left(\frac{m}{g_1g_2}\right) \ll 1.$$

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Theorem (.C, Chaubey, 2024)

Let $\nu \geq 2$ and let $\mathbf{c} = (c_1, \cdots, c_\nu) \in \mathbb{Z}^\nu$ be a fixed vector and $P(x) = x\mathcal{P}'(x)$, where $\mathcal{P}'(x) = c_{\nu-1}x^{\nu-1} + \cdots + c_2 x + c_1$. The limiting pair correlation measure of the sequence $(\mathcal{F}_{Q,P})_Q$ under the GRH exists and is given by

$$\mathcal{S}(\Lambda) \ll \frac{1}{\beta_P^{1+\epsilon}} \int_0^\Lambda \frac{1}{\lambda^{1-\epsilon}} \sum_{1 \leq m < \frac{2\Lambda}{\beta_P}} h_2(m) \log\left(\frac{2\lambda}{m\beta_P}\right) d\lambda,$$

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for any $\Lambda > 0$, *where* $\beta_P = \prod_p \left(1 - \frac{f_P(p)}{p^2}\right)$,
$f_P(p) = |\{1 \leq d \leq p : P(d) \equiv 0 \pmod{p}\}|$ *and* $h_2(m) = \frac{\sigma(m)}{m^{1+\epsilon}}$.

Key Lemmas

Lemma

Let r be an integer, and c_1, c_2 be positive integers and set $P(x) = c_2x^2 + c_1x$, then for any $\epsilon > 0$ under the GRH, we have

$$\sum_{\gamma \in \mathcal{F}_{Q,P}} e(r\gamma) \ll (c_1c_2)^{\epsilon} Q^{1+\epsilon} \sum_{g \mid c_1} \frac{1}{g^{1+\epsilon}} \sum_{\substack{q \leq \frac{Q}{g} \\ q \mid r}} \frac{1}{q^{\epsilon}},$$

where $e(x) = \exp(2\pi i x)$.

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where $e(x) = \exp(2\pi ix)$.

Lemma

Let $q \geq 2, t \in \mathbb{Z}$*, then for every* $\epsilon > 0$ *under the GRH, we have*

$$\sum_{\substack{n \leq z \\ \gcd(n,q)=1}} \mu(n) e\left(\frac{t\bar{n}_q}{q}\right) \ll z^{\frac{3}{4}+\epsilon}(\tau(q))^2.$$

Fractions with prime denominators

Let $\mathbf{c} = (c_1, \cdots, c_n) \in \mathbb{Z}^n$ be a fixed vector and $P(x) = c_n x^n + \cdots + c_1 x$. Define

$$\mathscr{M}_{Q,P} := \left\{ \frac{a}{p} : 1 \leq a \leq p \leq Q,\ \gcd(P(a), p) = 1, p \text{ is prime} \right\}.$$

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Theorem (.C, Chaubey, 2024)

The limiting pair correlation function of the sequence $(\mathscr{M}_{Q,P})_{Q \in \mathbb{N}}$ exists as $Q \to \infty$ and is Poissonian.

Races for Farey fractions

We define

$$S_P(Q;q,l) := \sum_{\substack{n\leq Q \\ n\equiv l \pmod q}} \sum_{\substack{a\leq n \\ \gcd(P(a),n)=1}} 1.$$

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Theorem (.C, Chaubey, 2024)

Let $q \geq 2$ be an integer, assume Haselgrove's condition for the modulus q, let l_1, l_2 be positive integers such that $l_1 \not\equiv l_2 \pmod q$ and $(q, l_1 l_2) = 1$, and let $P(x) = c_\nu x^\nu + \cdots + c_1 x \in \mathbb{Z}[x]$. Then, the set of values of Q for which the difference $S_P(Q;q,l_1) - S_P(Q;q,l_2)$ is strictly positive and the set of values of Q for which the difference $S_P(Q;q,l_1) - S_P(Q;q,l_2)$ is strictly negative are unbounded.

Races for Farey fractions

Denote

$$A(Q) = S(Q; q, l_1) - S(Q; q, l_2) \pm cQ^{\frac{1}{2}-\epsilon}$$

Remark

We get a sequence $\{Q_i\}_{i=1}^{[\log T]}$ in the interval $(1, T]$ such that $sgnA(Q_i) \neq sgnA(Q_{i+1})$ and $|A(Q_i)| > Q_i^{1/2-\epsilon}$. Hence, $A(Q)$ has at least $\gg \log T$ oscillations of size $Q^{1/2-\epsilon}$, in the interval $(1, T]$.

Thank You!