The principal Chebotarev density theorem

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Overview

- The Chebotarev density theorem
- Hilbert exact sequences
- Principal density
- Non-splitting of HES
- Generalizations

Motivation

At the time of Gauss, it was already noticed that there are the "same number" of prime integers which satisfy $p \equiv 1 \mod 4$ and those which satisfy $p \equiv 3 \mod 4$.

This implies that as one considers larger and larger primes, the frequency of primes that completely split and those which are inert in $\mathbb{Z}[i]$ approaches $1/2$.

Set-up

Let K/k be a finite Galois extension of number fields, with Galois group $G = \mathrm{Gal}(K/k)$. Denote by $\mathcal{P}_K$ the set of prime ideals of the ring of integers $\mathcal{O}_K$. For a prime ideal $\mathfrak{P} \in \mathcal{P}_K$, let $N_{K/k}(\mathfrak{P})$ be the relative norm map. When $k = \mathbb{Q}$ we will instead write $N\mathfrak{P}$.

For $\mathfrak{p} \in \mathcal{P}_k$, and $\mathfrak{P} \in \mathcal{P}_K$ unramified over $\mathfrak{p}$, the Artin symbol $\left(\frac{K/k}{\mathfrak{P}}\right)$ is the unique $\sigma \in G$ such that for all $x \in K$, we have

$$\sigma(x) \equiv x^{N\mathfrak{p}} \mod \mathfrak{P}.$$

Frobenius classes

The values of $\left(\frac{K/k}{\mathfrak{P}}\right)$ for all $\mathfrak{P}$ lying over $\mathfrak{p}$ are all conjugate. We denote by $\left(\frac{K/k}{\mathfrak{p}}\right)$ the conjugacy class of $\left(\frac{K/k}{\mathfrak{P}}\right)$ for all $\mathfrak{P}$ lying above $\mathfrak{p}$ and call $\left(\frac{K/k}{\mathfrak{p}}\right)$ the Frobenius class associated to $\mathfrak{p}$.

In general, when G is abelian, $\left(\frac{K/k}{\mathfrak{p}}\right)$ is a single element.

The prime $\mathfrak{p}$ of k completely splits if and only if $\left(\frac{K/k}{\mathfrak{p}}\right) = 1$.

Example

Example: Let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_m)$, where ζ_m is a primitive mth root of unity.

- K is the cyclotomic extension of degree $\varphi(m)$
- $\mathrm{Gal}(K/k) \cong (\mathbb{Z}/m\mathbb{Z})^*$ via $\zeta_m \mapsto \zeta_m^i$
- p ramifies in $\mathbb{Q}(\zeta_m)$ iff $p|m$.

Let $\mathfrak{P}$ be a prime above p. Let $\sigma \in \mathrm{Gal}(K/k)$ satisfy

$$\sigma(x) \equiv x^p \mod \mathfrak{P}$$

for all $x \in K$, then we must have $\sigma(\zeta_m^k) \equiv \zeta_m^{kp} \mod \mathfrak{P}$ for all k. Thus, in this case $\left(\frac{K/k}{p}\right) = \overline{p}$, the class of p modulo m.

The Chebotarev Density Theorem

For a given conjugacy class C of G and $\mathfrak{p} \in \mathcal{P}_k$ unramified in K, let

$$\mathcal{P}_{k,C} := \left\{ \mathfrak{p} \in \mathcal{P}_k \mid \left(\frac{K/k}{\mathfrak{p}} \right) = C \right\}.$$

The natural density of $\mathcal{P}_{k,C}$ is

$$\mu_{K/k}(C) := \lim_{N \to \infty} \frac{\# \left\{ \mathfrak{p} \mid N\mathfrak{p} \leq N, \left(\frac{K/k}{\mathfrak{p}} \right) = C \right\}}{\# \left\{ \mathfrak{p} \mid N\mathfrak{p} \leq N \right\}}.$$

Chebotarev density theorem

With the notation above, we have

$$\mu_{K/k}(C) = \frac{|C|}{|G|}.$$

Applications of CDT

If we again consider cyclotomic extensions, we see the connection between Chebotarev's density theory and Dirichlet's theorem on primes in arithmetic progressions.

Let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_m)$ with $\mathrm{Gal}(K/k) \cong (\mathbb{Z}/m\mathbb{Z})^*$. In this case Chebotarev's theorem says the density of primes p such that $p \equiv a \mod m$ is $\frac{1}{\varphi(m)}$.

Theorem of Dirichlet

Let m be a positive integer. Then for each integer a with $\gcd(a, m) = 1$ the set of prime numbers p with $p \equiv a \mod m$ has density $1/\varphi(m)$.

Applications of CDT

Let's consider another application of Chebotarev's density theorem.

Fix a number field $K/\mathbb{Q}$. The ideal class group of K, is the quotient group

$$Cl_K := I_K/P_K$$

where I_K is the group of fractional ideals of $\mathcal{O}_K$ and P_K is the subgroup of principal ideals of $\mathcal{O}_K$. We call the order of Cl_K the class number of K, denoted h_K.

The class group measures the failure of unique factorization in $\mathcal{O}_K$.

Applications of CDT

The Hilbert class field of K, denoted H_K, is the maximal abelian unramified extension of K.

Class field theory gives us an isomorphism $Cl_K \simeq \mathrm{Gal}(H_K/K)$ and $[H_K : K] = h_K$.

The isomorphism $Cl_K \to \mathrm{Gal}(H_K/K)$ sends a prime $\mathfrak{p}$ to its associated Frobenius class. In particular, a prime is totally split in H_K/K if and only if it is principal.

The CDT implies totally split primes in H_K/K have density $\frac{1}{|\mathrm{Gal}(H_K/K)|} = \frac{1}{h_k}$.

Effective Version

Set

$$\pi_C(x, K/k) := \#\left\{\mathfrak{p} \mid \mathfrak{p} \text{ is unramified in } K\ ,\left(\frac{K/k}{\mathfrak{p}}\right) = C, N\mathfrak{p} \leq x\right\}.$$

The Chebotarev density theorem gives:

$$\pi_C(x, K/k) \sim \frac{|C|}{|G|}\int_2^x \frac{dt}{\log t} \sim \frac{|C|}{|G|}\frac{x}{\log x} \text{ as } x \to \infty.$$

Effective Version

Ideally, we could determine effectively a smallest value $\tilde{x}$ for which $\pi_C(x, K/k) > 0$ if $x \geq \tilde{x}$.

It is important to be able to compute a bound on $N\mathfrak{p}$ below which every conjugacy class is realized as the Frobenius class of some $\mathfrak{p}$.

Early proofs of the CDT either had error estimates which depended on k and K in unclear ways, or none at all.

Effective Version

An effective version was shown in 1977 (assuming GRH):

Theorem (Lagarias, Odlyzko 1977)

There exists an effectively computable absolute constant $c_0 \geq 0$ such that for any Galois extension K/k with $G = \mathrm{Gal}(K/k)$, then for any fixed conjugacy class $C \subseteq G$ and every $x \geq 2$,

$$\left|\pi_C(x, K/k) - \frac{|C|}{|G|}\int_2^x \frac{dt}{\log t}\right| \leq c_0 \left(\frac{|C|}{|G|} x^{1/2} \log(|\Delta_K| x^n)\right).$$

As a corollary, there exists an effectively computable c_1 such that for every conjugacy class C of G there exists an unramified prime ideal $\mathfrak{p}$ of k such that $\left(\frac{K/k}{\mathfrak{p}}\right) = C$ and

$$N_{k/\mathbb{Q}}(\mathfrak{p}) \leq c_1 (\log |\Delta_K|)^2 (\log\log |\Delta_K|)^4.$$

The Chebotarev Density Theorem

Given a finite Galois extension K/k with group G and some conjugacy class of $C \subset G$, the Chebotarev density theorem says the frequency of primes of k whose corresponding Frobenius class is equal to C is given by $|C|/|G|$.

We will soon give a refined version of the natural densities which occur in the CDT. From this definition we produce a method to understand a special short exact sequence called the Hilbert short exact sequence.

Hilbert Exact Sequence

Let K/k be a finite Galois extension of number fields. The Hilbert class field of K, H_K, is Galois over k, and there is a natural restriction map

$$\pi : \mathrm{Gal}(H_K/k) \to \mathrm{Gal}(K/k)$$
$$\tau \mapsto \tau|_K$$

with $\ker(\pi) \cong \mathrm{Gal}(H_K/K) \cong Cl_K$. So, we obtain the Hilbert short exact sequence (HES):

$$1 \to Cl_K \to \mathrm{Gal}(H_K/k) \xrightarrow{\pi} \mathrm{Gal}(K/k) \to 1.$$

Split extensions

A group extension $1 \to N \to E \xrightarrow{\pi} Q \to 1$ is split if one of these equivalent conditions hold

- There exists a morphism $s : Q \to E$ such that $\pi \circ s = id_Q$. In this case, we say s splits the extension and call s a splitting.
- E is a semi-direct product of the form $N \rtimes Q$.
- The corresponding class in $H^2(Q, N)$ is trivial.

Splitting of the HES

We are motivated by the question of whether or not the HES:

$$1 \to Cl_K \to \mathrm{Gal}(H_K/k) \xrightarrow{\pi} \mathrm{Gal}(K/k) \to 1$$

splits. This question has been investigated by several people.

It was originally believed that the HES always split when $k = \mathbb{Q}$. This was shown to be false by Wyman in 1973.

Wyman proved the HES does split when k has class number one and K/k is cyclic. In 1977 Gold found another proof of Wyman's result, which was improved by Cornell and Rosen.

Splitting of the HES

In 1988 Cornell and Rosen proved a necessary condition for the splitting of the HES.

In the case when K/k is abelian of odd degree, this necessary condition is equivalent to whether or not the Hasse norm theorem holds for K.

Therefore, the result of Cornell and Rosen implies that in the case when $\mathrm{Gal}(K/k)$ is not cyclic, it is unlikely that the HES will split.

Splitting of the HES

For a concrete extension K/k, it is difficult to determine if the HES splits because it depends on $H^2(\mathrm{Gal}(K/k), Cl_K)$.

Therefore, one of our main motivations is to determine an algorithm which can check whether or not the HES splits of certain K/k.

First Main Result

We say a prime $\mathfrak{p}$ of k principally realizes a conjugacy class $C \subset \mathrm{Gal}(K/k)$ if $\mathfrak{p}$ is unramified in K and

- $\left(\frac{K/k}{\mathfrak{p}}\right) = C$
- $\mathfrak{p}$ is a product of principal prime ideals in $\mathcal{O}_K$

Theorem (Duan, Ma, O., Wang 2021)

Fix a Galois extension K/k. There is an effective bound B_K, such that if any conjugacy class C of $\mathrm{Gal}(K/k)$ cannot be principally realized by at least one prime $\mathfrak{p}$ of k with $N_{k/\mathbb{Q}}(\mathfrak{p}) \leq B_K$, then the HES does not split.

In particular, under the assumption of GRH, one can take

$$B_K = (4h_K \log|\Delta_K| + 2.5 \cdot n \cdot h_K + 5)^2,$$

where $n = [K : \mathbb{Q}]$, $|\Delta_K|$ is the absolute discriminant of K and h_K is the class number of K.

Principal Density

The proof of the above theorem is dependent on a refinement of $\mu_{K/k}$, where we consider primes $\mathfrak{p}$ of k which *principally* realize a given conjugacy class.

We define:

$$\mu^1_{K/k}(C) := \lim_{N\to\infty} \frac{\#\left\{\mathfrak{p} \in \mathcal{P}_k \mid N\mathfrak{p} \leq N,\ \left(\frac{K/k}{\mathfrak{p}}\right) = C, \mathfrak{P} \text{ is principal}\right\}}{\#\{\mathfrak{p} \in \mathcal{P}_k \mid N\mathfrak{p} \leq N\}}$$

for every prime ideal $\mathfrak{P}$ of K lying above $\mathfrak{p}$.

Questions about $\mu^1_{K/k}(C)$

Questions:

- Is $\mu^1_{K/k}(C)$ well defined?
- Is $\mu^1_{K/k}(C) > 0$?
- Is there an explicit formula for $\mu^1_{K,k}(C)$?
- How is $\mu^1_{K/k}(C)$ related to the HES?

For ease in notation, we will write the HES as

$$1 \to Cl_K \to E \xrightarrow{\pi} G \to 1,$$

where $E = \mathrm{Gal}(H_K/k)$ and $G = \mathrm{Gal}(K/k)$.

$\mu^1_{K/k}(C)$ is well defined

Proposition (Duan, Ma, O., Wang 2021)

Let C be a conjugacy class of G and $d_G(C)$ the common order of the elements of C. The density $\mu^1_{K/k}(C)$ is well defined and

$$\mu^1_{K/k}(C) = \frac{|\{\sigma \in E \mid \pi(\sigma) \in C \text{ and } \sigma^{d_G(C)} = id_E\}|}{|E|}.$$

From this result we see that $\mu^1_{K/k}(C)$ depends on the union of conjugacy classes of E.

To see this, write

$$E = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_r.$$

Assume $\sigma \in C_1$ and check if σ satisfies $\pi(\sigma) \in C$ and $\sigma^{d_G(C)} = id_E$. If one $\sigma \in C_1$ satisfies these conditions, all $\sigma \in C_1$ will. So we can write the numerator as a union of conjugacy classes of E.

When is $\mu^1_{K/k}(C) > 0$?

Our next main result describes how the splitting of the HES can determine when $\mu^1_{K/k}(C) > 0$.

Proposition (Duan, Ma, O., Wang 2021)

If the Hilbert exact sequence

$$1 \to Cl_K \to E \xrightarrow{\pi} G \to 1$$

splits, then $\mu^1_{K/k}(C) > 0$ for every conjugacy class C.

The main step in the proof of this result is to show that for a fixed $C \subset G$, the density $\mu^1_{K/k}(C) > 0$ if and only if there exists an element $g \in C$ such that

$$1 \to Cl_K \to E_g \to \langle g \rangle \to 1$$

splits, where for $g \in G$ we denote by $E_g := \pi^{-1}(\langle g \rangle) \subset E$.

When is $\mu^1_{K/k}(C) > 0$?

Therefore, $\mu^1_{K/k}(C) > 0$ for all conjugacy classes if and only if for every maximal cyclic subgroup U of G,

$$1 \to Cl_K \to \pi^{-1}(U) \to U \to 1,$$

splits.

Running over all maximal cyclic subgroups U of G gives us the result.

Explicit formula

We would like an explicit formula for $\mu^1_{K/k}(C)$ for a fixed conjugacy class $C \subset G$.

Assume $\mu^1_{K/k}(C) > 0$. Fix an element $\sigma \in E$ such that $\pi(\sigma) \in C$ and $\sigma^{d_G(C)} = \mathrm{id}_E$. Define a group homomorphism

$$N_\sigma : Cl_K \to Cl_K$$

by

$$x \mapsto (x\sigma)^{d_G(C)}.$$

Lemma (Duan, Ma, O., Wang 2021)

With the notations above

$$\mu^1_{K/k}(C) = \frac{|C|}{|G|} \frac{|\ker(N_\sigma)|}{h_k}.$$

Genus field

Let $g = \pi(\sigma) \in G$, then $\langle g \rangle$ acts on K. Call $F := K^{\langle g \rangle}$, the fixed field of K by $\langle g \rangle$.

There exists an intermediate field $H_K \supset K_F \supset K$ such that K_F is maximal among all such possible intermediate fields which are abelian extensions over F.

We call K_F the genus field of K over F and $[K_F : K]$ the genus number of K over F.

Explicit formula

Recall,

$$\mu^1_{K/k}(C) = \frac{|C|}{|G|}\frac{|\ker(N_\sigma)|}{h_k}.$$

By the theory of Tate cohomology and Galois theory we have

$$\frac{|\ker(N_\sigma)|}{h_K} = \frac{|H^1(\langle g\rangle, Cl_K)|}{[K_F : K]}.$$

Explicit formula

Theorem (Duan, Ma, O., Wang 2021)

For every conjugacy class C of G such that $\mu^1_{K/k}(C) > 0$, let $\sigma \in \pi^{-1}(g) \subset E_g$ be an element of order $d_G(C)$ for some $g \in C$. Then,

$$\mu^1_{K,k}(C) = \frac{|C|}{|G|} \frac{|H^1(\langle g \rangle, Cl_K)|}{[K_F : K]}.$$

The case when $C = \{\mathrm{id}_G\}$ is of special interest.

A special case

If we take $C = \{id_G\}$, then $\mu^1_{K/k}(\mathrm{id}_G) = \frac{1}{|G|h_K}$ since in this case we have

- $H^1(\langle \mathrm{id}_G \rangle, Cl_K) = \mathrm{Hom}(id_G, Cl_K)$ which is trivial
- $F = K^{\mathrm{id}_G} = K$ and so $K_F = H_K$.

In other words, the probability of finding a prime ideal of k which splits principally in K is $\frac{1}{|G|h_K}$.

Effective Method

So far:

- If the HES splits for a Galois extension with group G, then $\mu^1_{K/k}(C) > 0$ for every conjugacy class $C \subset G$.
- $\mu^1_{K/k}(C)$ is dependent on the union of conjugacy classes of $\mathrm{Gal}(H_K/k)$.

Goal: Find a bound such that every conjugacy class of $\mathrm{Gal}(H_K/k)$ can be realized as the Frobenius class of at least one prime ideal $\mathfrak{p}$ of k unramified in H_K.

Effective Method

Theorem (Bach, Sorenson 1996)

Let L/k be a Galois extension of number fields, with $L \neq \mathbb{Q}$. Let Δ_L denote the discriminant of L, and $n = [L : \mathbb{Q}]$. Let $C \subset \mathrm{Gal}(L/k)$ be a conjugacy class. Assume GRH. Then there is an unramified prime ideal $\mathfrak{p}$ of k with $\left(\frac{L/k}{\mathfrak{p}}\right) = C$ satisfying

$$N\mathfrak{p} \leq (4 \log |\Delta_L| + 2.5n + 5)^2.$$

We are left finding/estimating the degree $[H_K : \mathbb{Q}]$ and the discriminant Δ_{H_K}. Since $[H_K : \mathbb{Q}] = h_K[K : \mathbb{Q}]$, we need only to estimate Δ_{H_K}.

Effective Method

To determine the discriminant, we compute the norm of the different, a fractional ideal in the ring of integers of H_K. In our case, $\Delta_{H_K} = \Delta_K^{h_K}$.

Applying this to the result of Bach & Sorenson with $L = H_K$, we obtain:

Theorem (Duan, Ma, O., Wang 2021)

Let K/k be a Galois extension. Assuming GRH, take

$$B_K = (4h_K \log|\Delta_K| + 2.5 \cdot n \cdot h_K + 5)^2. \tag{1}$$

Then a conjugacy class $C \subset \mathrm{Gal}(K/k)$ satisfies $\mu^1_{K/k}(C) > 0$ if and only if there exists an unramified prime ideal $\mathfrak{p}$ of k with $N\mathfrak{p} \leq B_K$, and $\mathfrak{p}$ principally realizes C.

Testing the non-splitting of a HES

In particular, if the associated Hilbert exact sequence splits, then every conjugacy class C can be realized as the Frobenius class by at least one prime ideal $\mathfrak{p}$ as above.

Example: Let $K = \mathbb{Q}(\sqrt{-3}, \sqrt{13})$.

- K is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- $Cl_K = \mathbb{Z}/2\mathbb{Z}$ so $h_K = 2$
- It can be checked that $|\Delta_K| = 1521$.

One of the three quadratic subfields of K is $L = \mathbb{Q}(\sqrt{-3 \times 13})$. In order to show the HES does not split in this case, we need to find a conjugacy class C which can not be principally realized by any unramified prime in $\mathbb{Q}$.

Testing the non-splitting of a HES

Assume σ generates $\mathrm{Gal}(K/L)$ and take $C = \{\sigma\}$ to be the corresponding conjugacy class. Then the sub exact sequence

$$1 \to Cl_K \to E_\sigma \to \langle\sigma\rangle \to 1$$

splits if and only if one can find an unramified prime integer p such that

- p factors principally in K.
- p totally split in L; this guarantees that $\left(\frac{K/\mathbb{Q}}{p}\right) \in \mathrm{Gal}(K/L)$
- p is not totally split in K; this guarantees that $\left(\frac{K/\mathbb{Q}}{p}\right) \in C$

Testing the non-splitting of a HES

By the previous theorem, if such a p exists, it can be found under

$$\begin{aligned} B_K &= (4h_K \log|\Delta_K| + 2.5 \cdot n \cdot h_K + 5)^2 \\ &= (4 \times 2 \times \log|1521| + 2.5 \times 4 \times 2 + 5)^2 < 6992. \end{aligned}$$

One can verify with the help of a computer that no such prime integer exists. So, $\mu^1_{K/k}(\sigma) = 0$ and the associated HES does not split.

Principal density

The principal density gives us:

- a method for testing the non-splitting of the HES
- a way of "computing" the class number of a number field

Is there a way to generalize the notion of the principal density?

A Generalized Version

For any unramified $\mathfrak{p}$ in k lying below a prime $\mathfrak{P}$ in K we can define the K/k-principal order of $\mathfrak{p}$ to be the smallest positive integer $n_{K/k,\mathfrak{p}}$ such that $\mathfrak{P}^{n_{K/k,\mathfrak{p}}}$ is principal in K.

A Generalized Version

We can now consider the following density for every positive integer m:

$$\mu_{K/k}^m(C) := \lim_{N\to\infty} \frac{\#\left\{\mathfrak{p}\in\mathcal{P}_k \mid N\mathfrak{p}\le N,\ \left(\frac{K/k}{\mathfrak{p}}\right)=C, n_{K/k,\mathfrak{p}}|m\right\}}{\#\{\mathfrak{p}\in\mathcal{P}_k \mid N\mathfrak{p}\le N\}}.$$

For each positive integer m we can also define

$$\theta_{K/k}^m(C) := \lim_{N\to\infty} \frac{\#\left\{\mathfrak{p}\in\mathcal{P}_k \mid N\mathfrak{p}\le N,\ \left(\frac{K/k}{\mathfrak{p}}\right)=C, n_{K/k,\mathfrak{p}}=m\right\}}{\#\{\mathfrak{p}\in\mathcal{P}_k \mid N\mathfrak{p}\le N\}}.$$

Facts about $\mu^m_{K/k}$

Lemma (Duan, Ma, O., Wang 2021)

Let K/k be a Galois extension of number fields with $G = \mathrm{Gal}(K/k)$.

- For every conjugacy class $C \subset G$ and every positive integer m, the density $\mu^m_{K/k}(C)$ is well defined.
- $\mu^m_{K/k}(C) > 0$ for all conjugacy classes if and only if for every maximal cyclic subgroup U of G there exists a divisor i_U of m such that

$$1 \to Cl_K/Cl^0_K[i_U] \to \pi^{-1}(U)/Cl^0_K[i_U] \to U \to 1$$

exists and splits, where $Cl^0_K[n]$ denotes the subgroup of Cl_K generated by the elements of order exactly n.

Explicit formula

We define a homomorphism $N_{\sigma,m}$ as before. Since there is an element σ such that $\sigma^{d_G(C)} = id_E$ we have

$$N_{\sigma,m} = (N_{\sigma,1})^m : x \mapsto (N_{\sigma,1}(x))^m.$$

Then

$$\mu^m_{K/k}(C) = \frac{|C||\ker(N_\sigma, m)|}{|G|h_K}.$$

Since

$$\ker(N_{\sigma,m})/\ker(N_{\sigma,1}) = (Cl_K/\ker(\mathrm{N}_{\sigma,1}))[m],$$

we obtain the following result:

Corollary (Duan, Ma, O., Wang 2021)

With all the notations above, if $\mu^1_{K/k}(C) > 0$, we have

$$\mu^m_{K/k}(C) = \frac{|C|}{|G|}\frac{|H^1(\langle\sigma\rangle, Cl_K)|}{[K_F : K]}|(Cl_K/\ker(\mathrm{N}_{\sigma,1}))[m]|.$$

A special case

Take $C = \{id_G\}$, in this case we have

$$\ker(N_{\mathrm{id}_E,m}) = \{x \in Cl_K \mid x^m = \mathrm{id}_E\} = Cl_K[m].$$

Corollary (Duan, Ma, O., Wang 2021)

Taking $C = \{\mathrm{id}_G\}$ to be the trivial conjugacy class in G, for every prime integer p and every positive integer r, we have

$$\frac{\mu_{K/k}^{p^r}(\{\mathrm{id}_G\})}{\mu_{K/k}^{p^{r-1}}(\{\mathrm{id}_G\})} = \frac{|Cl_K[p^r]|}{|Cl_K[p^{r-1}]|}.$$

This results tells us that one can see the structure of Cl_K by the densities $\mu_{K/k}^m(\{\mathrm{id}_G\})$ as m varies!

Thank you!