The question of q, a look at the interplay of number theory and ergodic theory in continued fractions

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No, not that Q

The frame of mind for this talk

Usually, when a student first encounters the intersection of number theory, ergodic theory, and symbolic dynamics, they look at base-b expansions.

Base-b expansions are GREAT!

Almost too great!

So then one wanders into a more complicated expansions, such as generalized Lüroth series, β-expansions, or continued fraction (CF) expansions.

The latter will be the focus of our talk. We will look at a number of different problems and see how we overcome the unique challenges posed here.

Some background

Every $x \in [0, 1)$ can be expressed as a (regular) continued fraction expansion:

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

with $a_i \in \mathbb{N}$.

This expansion is infinite and unique for all irrational numbers. It is finite for all non-zero rational numbers, with two possible expansions.

For compact notation, we will often write $x = [a_1, a_2, a_3, \ldots]$.

Some background

The Gauss map $T : [0,1) \to [0,1)$ given by

$$Tx = \begin{cases} \frac{1}{x} \mod 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

acts as a forward shift on the CF digits of x. That is,

$$T[a_1, a_2, a_3, \dots] = [a_2, a_3, \dots]$$

T does not preserve Lebesgue measure, but does preserve the Gauss measure

$$\mu(A) = \int_A \frac{dx}{(1+x)\log 2}.$$

T is ergodic, and in fact, exact, with respect to μ.

A first application of the pointwise ergodic theorem

Given $k \in \mathbb{N}$ let C_k be the corresponding cylinder set:

$$C_k = \{x \in [0,1) : a_1(x) = k\}.$$

Then $C_k = [1/(k+1), 1/k)$. Importantly $\mu(C_k) \asymp k^{-2}$.

The ergodicity of T tells us that for almost every $x \in [0,1)$...

- the digit 1 appears with limiting frequency $\log(3/2)/\log(2)$.
- digits larger than 5 appear with limiting frequency $\log(7/6)/\log(2)$.

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- the digit 1 appears with limiting frequency $\log(3/2)/\log(2)$.
- digits larger than 5 appear with limiting frequency $\log(7/6)/\log(2)$.
- even digits appear with limiting frequency $\log(4/\pi)/\log(2)$.

More applications of the pointwise ergodic theorem

Also a fairly direct consequence of pointwise ergodic theorem:

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} \to \infty, \ \mu - \text{a.e.}$$

This follows because

$$a_k(x) = a_1(T^{k-1}x) = \sum_k k \cdot 1_{C_k}(T^{k-1}x)$$

and

$$\int_0^1 \frac{\sum_k k \cdot 1_{C_k}(x)}{(1+x)\log 2}\, dx = \sum_k k\mu(C_k) \asymp \sum_k k \cdot \frac{1}{k^2}$$

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This is rather unsatisfying. We'd like to know a more quantitative result. *How fast* does this average go to infinity?

A metric lemma

Lemma

Let $\phi : \mathbb{N} \to \mathbb{R}_{>0}$. *For almost all* $x \in [0,1)$, *there are infinitely many* n *with*

$$a_n(x) \geq \phi(n)$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$$

converges.

So in particular $a_n \geq n \log n$ should occur infinitely often for a.a. x, but $a_n \geq n \log^2 n$ should not.

Proving the metric lemma

As for a proof, here's the central idea:

$$\{x \in [0,1) : a_1(x) \geq \phi(1)\} = [0, 1/\phi(1))$$

and similarly

$$\mu(\{x \in [0,1) : a_k(x) \geq \phi(k)\} \asymp \frac{1}{\phi(k)}$$

From here, it's a Borel-Cantelli-type argument.

A heuristic

Because of the lemma, we expect that $a_1, \ldots, a_n$ are at most $n \log^2 n$ almost everywhere.

So therefore, we might expect that

$$
\begin{aligned}
a_1 + a_2 + \cdots + a_n &\approx n \int_0^1 \frac{\sum_{k \le n \log^2 n} k 1_{C_k}(x)}{(1+x)\log 2} \, dx \\
&= \frac{n}{\log 2} \sum_{k \le n \log^2 n} k \cdot \mu(C_k) \asymp \frac{n}{\log 2} \sum_{k \le n \log^2 n} \frac{1}{k} \\
&\asymp \frac{n \log n}{\log 2}.
\end{aligned}
$$

And indeed Khinchin was able to make this precise although he only showed the convergence in measure, not almost everywhere.

Theorem (Diamond-Vaaler, 1986)

For almost all $x \in [0,1)$ we have

$$a_1 + a_2 + \cdots + a_n = \frac{n \log n}{\log 2}(1 + o(1)) + \theta \cdot \max_{k \leq n} a_k,$$

where $\theta \in [0,1]$.

Basically this result is proved by doing the previous argument just a little more carefully.

First, consider when a_n and a_m are both larger than $N(\log N)^{3/4}$ with $n, m \leq N$. For almost all $x \in [0,1)$, this should only happen for finitely many N's.

Second, instead of $a_1 + \cdots + a_n$, study $a_1^* + \cdots + a_n^*$ where $a_i^* = a_n$ unless $a_i^* \geq N(\log N)^{3/4}$, in which case $a_i^* = 0$. The argument of the previous slides can be made more precise by using $\sum a_i^*$.

From hard problem to easy problem

Somewhat ironically, the geodesic mean of the digits is easy to calculate.

Instead of

$$(a_1 a_2 \dots a_n)^{1/n}$$

we look at the logarithm of this, which is

$$\frac{1}{n} \sum \log(a_i),$$

and the corresponding function

$$\sum \log(k) \cdot 1_{C_k}(x)$$

is integrable.

Some more background

Suppose $x = [a_1, a_2, a_3, \dots]$, then we define the nth convergent by

$$\frac{p_n}{q_n} = [a_1, a_2, \dots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n}}}.$$

We have the following matrix relation for convergents:

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

In particular $q_0 = 1$, $q_1 = a_1$ and $q_n = a_n q_{n-1} + q_{n-2}$ if $n \geq 2$.

More background

We can derive some more facts from the matrix relation:

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

For starters, by taking determinants of each side, we see that $p_{n-1}q_n - p_n q_{n-1} = (-1)^n$, and thus that

$$p_n q_{n-1} = (-1)^{n+1} \pmod{q_n}.$$

Also by taking transposes of the above matrix relation, we get that

$$\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}$$

and thus $q_{n-1}/q_n = [a_n, a_{n-1}, \dots, a_1]$.

Why Q?

It turns out that q_n is everywhere in CF theory. It's arguably *the* critical statistic about a CF expansion.

The q_n's essentially measure how far x is from its convergent p_n/q_n:

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}.$$

A natural first question about q_n

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Recalling that $q_n = a_n q_{n-1} + q_{n-2}$, and that $a_n \geq 1$. We get that q_n should grow at least exponentially. Specifically, we always have that

$$q_n \geq q_{n-1} + q_{n-2}$$

so that the q_n's grow at least as fast as the Fibonacci sequence.

Some initial analysis

Let's look at the relation

$$q_n = a_n q_{n-1} + q_{n-2}$$

again.

On the one hand, we have

$$q_n \geq a_n q_{n-1}$$

so that

$$q_n \geq a_n q_{n-1} \geq a_n a_{n-1} q_{n-2} \geq \cdots \geq a_n a_{n-1} \cdots a_1$$

Some initial analysis

On the other hand, we have

$$q_n \leq a_n q_{n-1} + q_{n-1} = (a_n + 1) q_{n-1}$$

and so

$$\begin{aligned} q_n &\leq (a_n + 1) q_{n-1} \leq (a_n + 1)(a_{n-1} + 1) q_{n-2} \\ &\leq \cdots \leq (a_n + 1)(a_{n-1} + 1) \ldots (a_1 + 1) \end{aligned}$$

Some initial analysis

Therefore we have

$$\sum_{i=1}^{n} \log a_i \leq \log q_n \leq \sum_{i=1}^{n} \log(a_i + 1)$$

and so

$$\frac{1}{n}\sum_{i=1}^{n} \log a_i \leq \frac{1}{n}\log q_n \leq \frac{1}{n}\sum_{i=1}^{n} \log(a_i + 1)$$

But we've already seen, by the pointwise ergodic theorem, that the sums on the left and right side converge for almost all x.

Beginner versus expert

The previous slide tells us that there exist $1 \leq \lambda \leq \Lambda$ with

$$\lambda^n \leq q_n \leq \Lambda^n$$

for almost all x.

But the stronger result (due to Khinchin) is

Theorem

For almost all x, we have

$$\lim_{n\to\infty} \frac{\log q_n}{n} = \frac{\pi^2}{12\log 2}.$$

Proving Khinchin's result

As the result implies, we should look at $\frac{1}{n} \log q_n$. If we expect q_n to grow exponentially, this should be constant.

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$$\frac{1}{n}\log q_n = -\frac{1}{n}\sum_{k=1}^{n}\log(q_{k-1}/q_k)$$

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Also recall that $q_{k-1}/q_k = [a_k, a_{k-1}, \dots, a_1]$. This suggests we need some kind of "reverse" continued fraction.

The natural extension of T

Let $\tilde{T} : [0,1)^2 \to [0,1)^2$ act by

$$\tilde{T}(x,y) = \left(Tx, \frac{1}{a_1(x) + y} \right).$$

This acts as a bijective digit shift:

$$\tilde{T}([a_1, a_2, \ldots,], [a_0, a_{-1}, \ldots]) = ([a_2, a_3, \ldots], [a_1, a_0, \ldots]).$$

This is once again ergodic and preserves the measure

$$\tilde{\mu}(A) = \int_A \frac{dx\ dy}{(1+xy)^2 \log 2}$$

The natural extension of T

In this natural extension we have that

$$\tilde{T}^n(x,0) = \left(T^n x, \frac{q_{n-1}}{q_n} \right).$$

So we can measure

$$\frac{1}{n}\log q_n = -\frac{1}{n}\sum_{k=1}^{n} \log(q_{k-1}/q_k)$$

by looking at an ergodic average over $f(x,y) = -\log(y)$.

Sure enough

$$\int_{[0,1)^2} \frac{-\log y}{(1+xy)^2 \log 2} dx\ dy = \frac{\pi^2}{12 \log 2}$$

Other facts about q_n

Now that we know how fast q_n grows, what else could we say about it?

Theorem (Moeckel)

For almost every x, we have that each of

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod 2$$

are each equally likely.

And equivalent versions for other moduli exist.

Proving Moeckel

How could one prove this? Many different ways!

Moeckel's original proof (as well as work of Nakanishi) relied on hyperbolic geometry. Here, the ergodicity of geodesic flow on certain surfaces is well known. The "modularity" of p_n/q_n can then be connected to the geodesic visiting certain cusps.

Jager and Liardet instead formed a skew product over the Gauss map T:

$$\hat{T}(x, M) = \left(Tx, M \begin{pmatrix} 0 & 1 \\ 1 & a_1(x) \end{pmatrix} \mod n \right).$$

Ergodicity of $\hat{T}$ is tricky, but follows straightforward methods.

Recently, Fisher and Schmidt (2014) combined the best of both methods.

A less ergodic ordering

From an ergodic perspective, we want to arrange rationals p/q in order by their number of CF digits...

Which is a very unusual way to arrange things.

What if we want to just study all fractions a/q with a fixed denominator q?

This is actually really hard and we'll come back to it.

The mathematics named after a geologist

For a simpler problem, let's consider the Farey fractions:

$$\mathcal{F}_Q = \left\{ \frac{a}{q} \in [0,1] : q \leq Q \right\}.$$

Theorem (Baladi-Vallee, 2006)

If $L(a/q)$ denotes the length of the CF expansion of a/q, then when considering $a/q \in \mathcal{F}_Q$ as $Q \to \infty$, $L(a/q)$ has a Gaussian distribution.

This is a big, deep result. We will not even attempt to prove it here.

A simpler version

First, we need a metric ergodic theorem (first due to Philipp?): we get that for any $\epsilon > 0$ and any positive integer N, we have that

$$\mu\left(\left\{x \in [0,1) \setminus \mathbb{Q} : \left|\frac{\log q_N(x)}{N} - \frac{\pi^2}{12\log 2}\right| > \epsilon\right\}\right) = O\left(\frac{1}{N}\right).$$

Functionally, the proof of this result relies on bounding

$$\int \left(\frac{1}{N}\sum_{i=1}^{N} -\log(q_{i-1}(x)/q_i(x)) - \frac{\pi^2}{12\log 2}\right)^2 \, dx.$$

However, to evaluate part of this integral, we already have to know that T is mixing with a good error term!

A simpler version

$$\mu\left(\left\{x \in [0,1) \setminus \mathbb{Q} : \left|\frac{\log q_N(x)}{N} - \frac{\pi^2}{12\log 2}\right| > \epsilon\right\}\right) = O\left(\frac{1}{N}\right).$$

This is a metric result, not a result about $\mathcal{F}_Q$, but it turns out its relatively easy to go back and forth between them. $\mathcal{F}_Q$ is quite well distributed across $[0,1]$.

So you can get results about almost all $a/q \in \mathcal{F}_Q$ have typical properties.

An even more natural ordering

The Farey fractions $\mathcal{F}_Q$ are a well-studied object, but perhaps an even more natural object of study are fractions with a fixed denominator: $\mathcal{M}_q = \{a/q : (a,q) = 1\}$.

Avdeeva and Bykovskii (2013) showed that $L(a,q)$ over $\mathcal{M}_q$ behaves *similarly* to having a Gaussian distribution: the average and variance one might expect, but instead of an exact Gaussian distribution, it simply has "similar tails."

How to prove this?

A surprising connection

The idea of Avdeeva and Bykovskii is to connect $\mathcal{M}_q$ to $\mathcal{F}_{\sqrt{q}}$.

Every $a/q \in \mathcal{M}_q$ has a nearby fraction $m/n \in \mathcal{F}_{\sqrt{q}}$, and the CF expansion of m/n is – essentially – the first half of the CF expansion of a/q.

Alternately, we could look at a^*/q, where $a^*a = 1 \pmod{q}$. The CF expansion of a^*/q is (almost) the reverse of the CF expansion of a/q. There must again be a nearby $m^*/n^* \in \mathcal{F}_{\sqrt{q}}$ whose CF expansion will be essentially the latter half of the CF expansion of a/q (in reverse).

Together the CF expansions of m/n and m^*/n^* won't, in general, give the CF expansion of a/q perfectly, but they can only miss by a few digits.

A surprising connection

So to each $a/q \in \mathcal{M}_q$, we can associate a pair of fractions $(m/n, m^*/n^*)$ in $\mathcal{F}_{\sqrt{q}}$. This is almost a bijection.

So in total, you can do this splitting argument and bootstrap known results about $\mathcal{F}_{\sqrt{q}}$ to new results about $\mathcal{M}_q$. While we've described this for measuring lengths of CF expansions, it works for a variety of measurements and statistics.

A renewal-type theorem

Theorem (Sinai-Ulcigrai, 2008)

For $R > 0$, let $n_R = \min\{n \in \mathbb{N} : q_n > R\}$. Then q_{n_R}/R and the digits a_{n_R-k} for $0 \leq k \leq K$ have a joint limiting distribution as R goes to infinity.

Proof: about 10 pages of pretty intense ergodic theory. The limiting distribution is also not calculated.

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Theorem (Ustinov, 2008)

All of q_{n_R-1}/R, R/q_{n_R}, and the digits a_{n_R+k} for $|k| \leq K$ have a joint limiting distribution as R goes to infinity.

Proof: about 1 page of number theory. The limiting distribution is given explicitly.

A renewal-type theorem

How did Ustinov accomplish this?

First, he made use of a matrix of the form $\begin{pmatrix} P & P' \\ Q & Q' \end{pmatrix}$ as a stand-in for matrices of the form $\begin{pmatrix} p_{n_R-1} & p_{n_R} \\ q_{n_R-1} & q_{n_R} \end{pmatrix}$.

Importantly, we have that

$$\frac{q_{n_R-1}}{q_{n_R}} = [a_{n_R}, a_{n_R-1}, \dots, a_1]$$

$$\frac{q_{n_R}x - p_{n_R}}{-q_{n_R-1}x - p_{n_R-1}} = [a_{n_R+1}, a_{n_R+2}, a_{n_R+3}, \dots]$$

So this matrix will also largely encode nearby digits as well.

You end up with a sum over a big collection of matrices of some relatively simple functions.

Thank you for attending!