Joel Feldman, winner of the 2007 CRM-Fields-PIMS Prize

by David Brydges (University of British Columbia)

My first encounter with the work of Joel Feldman came when I was a graduate student in 1970s. In those days the program to construct quantum field theories (QFT) was in full flight. The construction of two spacetime dimensional QFT had been successful and even the formidable divergences of three dimensional QFT were under siege by a machine invented by Glimm and Jaffe [1] and known as "the phase-cell expansion". Joel Feldman was one of a very small number of people who clearly understood the principles of this technique and I was reading his papers almost in preference to the papers of the masters because, then as now, his papers were models of clarity. In 1976 Feldman and Osterwalder [2] were finally able to prove that a candidate three-dimensional quantum field theory known as ϕ^4_3 was indeed a quantum field theory, that is, it satisfied the Wightman axioms. As for four spacetime dimensions, my adviser said to me "it may be 100 years before we understand four dimensions". Since then I have watched in astonishment as Joel Feldman and his collaborators have gradually extended the scope of the phase cell expansion to such an extent that their recent work on the Fermi surface problem overcomes a problem that is much more than the equal of the particular difficulties that made my adviser so pessimistic 30 years ago. Unfortunately, he may yet be right in that the original goal of existence of a four-dimensional theory remains unsolved. However, the obstruction is no longer the "short distance divergences" that seemed so hard in 1976.

To describe Joel Feldman's place in all of this and give a sense of the accomplishments that the CRM-Fields-PIMS prize honours, I will have to give a little background on QFT and the phase cell expansion. If another person were to write this article they might emphasize other parts of Feldman's work such as his work on infinite genus Riemann surfaces [3] but I take most pleasure in his work in QFT and condensed matter. In one other respect at least this description does a bad job. To keep it short the contributions of theoretical physics are not properly described.

The axioms of quantum field theory (QFT) are surprisingly simple. To each bounded open subset of spacetime is associated an algebra of operators on a Hilbert space. These operators represent quantities one can measure, such as the energy of the field inside the set. The structure of spacetime is encoded functorially by requiring that when two open sets in spacetime are related by a morphism the corresponding algebras of operators must be related by a corresponding morphism. Yet this simplicity conceals one of the most beautiful and difficult structures ever encountered in mathematics. We now hear about it all the time in differential geometry and topology, but for analysts it may offer the greatest challenge of all. All the strange arguments based on the "functional integral" will remain fringe intuition for conjectures proved some other way for as long as this challenge is unmet. Brownian motion, the pride and joy of probability, is a quantum field theory on a one-dimensional Euclidean spacetime. Among the other quantum field theories there will be other treasures. In fact SLE and conformal QFT in two dimensions is an unfolding example.

Phase space is the Cartesian product of space, $\mathbb{R}^d$ with momentum space, $\mathbb{R}^d$. As in the theory of pseudo-differential operators, a phase-cell is a box which is consistent with Fourier analysis and the uncertainty principle in that the sides Δx in spatial $\mathbb{R}^d$ and Δp in momentum $\mathbb{R}^d$ satisfy $\Delta x \Delta p = 1$. Divergences in quantum field theory arise because the field has fluctuations on all scales. If fluctuations are decomposed into fluctuations localised in phase-cells then there is roughly only one degree of freedom of fluctuation per phase-cell. The divergences of QFT are now lurking in the infinity of phase cells, but this reorganisation uncovers a compensating property of approximate independence. The phase-cell expansion implements the idea that the functional integral of quantum field theory is approximately the product over phase cells of integrations over fluctuations in phase cells. According to the phase cell expansion a quantum field theory functional integral can be understood as a convergent sum of corrections to this oversimplified picture. In this expansion the divergences of QFT appear in combinations where they cancel each other. The term "divergent" means that a parameter is introduced. It is called an "ultraviolet cutoff" and it specifies a small length at which fluctuations are artificially suppressed. The suppression destroys at least one of the axioms of QFT, but the phase cell expansion is uniform in this parameter and so the limit as the ultraviolet cutoff is taken to zero becomes feasible and the axioms are restored in the limit.

The Fermi surface problem arises in condensed matter physics. The object is to understand the collective behaviour of a large number of electrons moving in a crystal. The term collective behaviour is a signal that this is in the same class of problems as QFT: the many degrees of freedom of the electrons give rise to new phenomena such as superconductivity that are in the domain of QFT. If interactions between electrons are neglected then we can first focus on a single electron moving in the crystal. According to textbook quantum mechanics there are allowed energies which are functions of the momentum of the electron. Then the ground state for N noninteracting electrons is such that the lowest single electron energies are fully packed up to a sharp threshold called the Fermi surface and all higher energies are unoccupied. This is because electrons are Fermions and the antisymmetry of N particle Fermionic states under interchange of particle coordinates forbids multiple occupation of any single particle state so the electrons fill the levels like water in a bath. The surface can be visualised as enclosing a part of momentum space which is fully occupied. What happens to this picture if interactions between electrons are not neglected? Does a jump in the density of occupied states as a function of momentum persist or does it smooth out? The work of Feldman, Knörrer and Trubowitz answers this question, for a certain class of models, in the sequence of papers [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] posted on Feldman's homepage. Their work shows that if the Fermi surface for noninteracting electrons in a two dimensional system satisfy certain conditions then the result of turning on a small interaction is a deformation of the noninteracting surface as opposed to its destruction. Furthermore the renormalised perturbation theory is not merely finite term by term but convergent.

One of the major new ideas in these papers is to generalise the phase cell expansion to allow curvilinear phase cells that follow the geometry of the Fermi surface and refine as the surface is approached. A single electron, not interacting with others, propagates according to a partial differential equation for which the Fermi surface is characteristic. In the dynamics of many noninteracting electrons phase cells near this characteristic surface harbour the large fluctuations. However with this geometry in the phase cell decomposition the interactions between fluctuations have the necessary independence properties as one gets closer to the surface.

As in QFT there are divergences in naive perturbation theory. These arise because the Fermi surface of the noninteracting electron system is not the same as the Fermi surface of the interacting system. Renormalisation means making the expansion about a new noninteracting system whose Fermi surface is the same as the interacting Fermi surface.

A Cooper pair consists of two particles, each of whose momenta lie on the Fermi surface, and whose total momentum is very small. If the shape of the surface permits such pairs then the interaction between these pairs becomes so huge as their total momentum approaches zero that new divergences appear in the perturbation theory. This is again a signal that the model is being approximated by the wrong noninteracting model. In these Fermi surface papers this phenomenon is exactly what the hypotheses are ruling out. Conjecturally, if they are not ruled out, the Cooper pairs dominate the long distance structure of the theory, the Fermi surface is destroyed and the system is a superconductor.

In the 1980s, in the work of Balaban and others, it became clear that taking the limit as the ultraviolet cutoff tends to zero is possible, but there is another obstruction to existence of four-dimensional QFT in the sense that is specified in the Clay Institute problem. One must also understand the role of fluctuations at very long scales. The phenomenon of superconductivity in the many electron problem is very relevant to what is conjectured to happen in four-dimensional gauge QFT: superconductors expel magnetic field by setting up a flow of electrons to create an opposite magnetic field. This is called the Meissner effect. In some versions of this effect instead of complete expulsion, which is energetically costly, a thin tube reverts to the nonsuperconducting phase so as to permit the magnetic field but only in the tube. If two magnetic monopoles were placed in such a system there would be a flux tube joining them whose cost in energy grows linearly with length. Therefore the monopoles would be "confined" to stay close to each other by the flux tube. In four-dimensional QFT a similar mechanism is expected to confine quarks. A more detailed grip on this type of mechanism is needed to understand four-dimensional gauge QFT. The Fermi surface work has built part of the infrastructure for this problem. In fact, to prove existence of the Fermi surface the authors had to build an enormous infrastructure, including definitions, properties and estimates for integration over Grassmann algebras in order to express Fermions in the language of QFT, analysis and estimates of classes of ladder diagrams and the technology of phase cells adapted to the Fermi surface.

When I contemplate the range of phenomena that QFT rules it seems to me that Nature lives on the edge of what is possible and an extraordinary amount of preparation may be needed for us to follow it there. By gathering in one place complete proofs that end with a very strong control over perturbation theory for condensed matter, the expedition is well under way.

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