The Hardy–Littlewood prime tuple conjecture and Ramanujan sums

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Twin prime conjecture

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Conjecture 0.1

There are infinitely many twin primes or pairs of primes that differ by 2.

Hardy-Littlewood k-tuple conjecture

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- Let $a_1, \cdots, a_k$ be distinct integers, and $b(p)$ be the number of distinct residue classes (mod p) represented by a_i, $1 \leq i \leq k$.
- If $b(p) < p$ for every prime p, then

$$\#\{n \leq x : n + a_i \text{ are primes } \forall 1 \leq i \leq k\} \sim \mathfrak{S}(a_1, ..., a_k) \frac{x}{(\log x)^k},$$

where

$$\mathfrak{S}(a_1, ..., a_k) = \prod_p \left(1 - \frac{b(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-k},$$

and the product is over all primes p.

Equivalent form of k-tuple conjecture

- The prime k-tuple conjecture is equivalent to show

$$\sum_{n \leq x} \Lambda(n + a_1) \cdots \Lambda(n + a_k) \sim \mathfrak{S}(a_1, ..., a_k) x,$$

where "Λ" is the von Mangoldt function.

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

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- One can show

$$\sum_{n \leq x} \Lambda(n+a_1) \cdots \Lambda(n+a_k) \sim \sum_{n \leq x} \frac{\phi(n+a_1)}{n+a_1} \Lambda(n+a_1) \cdots \frac{\phi(n+a_k)}{n+a_k} \Lambda(n+a_k).$$

Ramanujan sums

- Hardy's formula

$$\frac{\phi(n)\Lambda(n)}{n} = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n).$$

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- $c_q(n)$ is called the Ramanujan sums and it is defined as:

$$c_q(n) := \sum_{\substack{j=0 \\ (j,q)=1}}^{q-1} e\left(\frac{jn}{q}\right) = \sum_{\substack{d|n \\ d|q}} d\mu\left(\frac{q}{d}\right),$$

where $e(x) = e^{2\pi ix}$.

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- Observe that $c_q(0) = \phi(q)$ and $c_q(1) = \mu(q)$.
- If $(q_1, q_2) = 1$, then

$$c_{q_1 q_2}(n) = c_{q_1}(n) c_{q_2}(n).$$

Heuristic derivation of k-tuple conjecture

- **Heuristically,**

$$\sum_{q_1,\cdots,q_k}^{\infty} \frac{\mu(q_1)\cdots\mu(q_k)}{\phi(q_1)\cdots\phi(q_k)} \sum_{n\leq x} c_{q_1}(n+a_1)\cdots c_{q_k}(n+a_k)$$
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- It is equivalent to show

$$\lim_{x\to\infty} \frac{1}{x} \sum_{q_1,\cdots,q_k}^{\infty} \frac{\mu(q_1)\cdots\mu(q_k)}{\phi(q_1)\cdots\phi(q_k)} \sum_{n\le x} c_{q_1}(n+a_1)\cdots c_{q_k}(n+a_k) = \mathfrak{S}(a_1, ..., a_k). \tag{1}$$

Orthogonality property of Ramanujan sums

Theorem 0.1 (Carmichael, 1932)

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} c_r(n) c_s(n+h) = \begin{cases} c_r(h) & \textit{if } r = s, \\ 0 & \textit{otherwise.} \end{cases}$$

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Proof.

We have

$$\sum_{n\le x} c_r(n)c_s(n+h) = \sum_{(a,r)=1} \sum_{(b,s)=1} e^{2\pi i h b/s} \sum_{n\le x} e^{2\pi i n(a/r+b/s)}.$$

The innermost sum is bounded unless $a/r + b/s$ is an integer. □

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In 1999, Gadiyar and Padma discovered a simple heuristic to derive the case $k = 2$.

Triple convolution of Ramanujan sums

Theorem 0.2 (Chaubey, G., Murty, 2023)

Let r, s, t be squarefree with $(a, r) = (b, s) = (c, t) = 1$. Then,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_r(n+k)c_s(n+h)c_t(n+j)$$
$$= \mathscr{K}_\Delta(h-k, j-k)c_U(h-j)c_V(j-k)c_W(h-k),$$

where $r = \Delta UV$, $s = \Delta UW$, and $t = \Delta VW$ with Δ, U, V, W all mutually coprime and c_U, c_V, c_W are Ramanujan sums and

$$\mathscr{K}_r(h, j) := \sum_{\substack{(b,r)=(c,r)=1 \\ (b+c,r)=1}} e^{2\pi i(hb+jc)/r}.$$

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Using Theorem 0.2, we derive a heuristic proof for the case $k = 3$.

Two variable variant of Ramanujan sums

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- We have the following orthogonality property

$$\lim_{\substack{x\to\infty\\y\to\infty}} \frac{1}{xy} \sum_{\substack{h\leq x\\j\leq y}} \mathscr{K}_r(h,j)\overline{K_s(h,j)} = f(r)\delta_{r,s},$$

where $\delta_{r,s}$ is the Kronecker delta function and

$$f(r) = \sum_{\substack{(b,r)=(c,r)=1\\(b+c,r)=1}} 1.$$

Higher convolutions of Ramanujan sums

- Assume that

$$f(q_1, ..., q_k) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} c_{q_1}(n + a_1) \cdots c_{q_k}(n + a_k).$$

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- From definition, we have

$$f(q_1, ..., q_k) = \lim_{x\to\infty} \frac{1}{x} \sum_{d_1|q_1,...,d_k|q_k} d_1\mu\left(\frac{q_1}{d_1}\right) \cdots d_k\mu\left(\frac{q_k}{d_k}\right) \sum_{\substack{n\leq x \\ d_1|a_1+n,\cdots,d_k|a_k+n}} 1$$

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$$= \lim_{x\to\infty} \frac{1}{x} \sum_{d_1|q_1,\dots,d_k|q_k} d_1\mu\left(\frac{q_1}{d_1}\right)\cdots d_k\mu\left(\frac{q_k}{d_k}\right) \sum_{\substack{n\le x \\ n\equiv -a_1 \mod d_1 \\ \vdots \\ n\equiv -a_k \mod d_k}} 1.$$

Generalized Chinese Remainder Theorem

Lemma 1

For a fixed set $T = \{a_1, \cdots, a_k\}$ *and* $d_1, \cdots, d_k \in \mathbb{Z}$*, the system*

$$\begin{aligned} x &\equiv a_1 \mod d_1 \\ &\vdots \\ x &\equiv a_k \mod d_k \end{aligned} \tag{3}$$

has a solution if and only if $(d_i, d_j) | (a_i - a_j)$ *for all* $1 \le i, j \le k$*. When the solution exists, it is unique modulo* $[d_1, \cdots, d_k]$*.*

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From now on, we will fix T and define a function

$$g(d_1, \cdots, d_k) := \begin{cases} 1 & \text{if (3) has a solution,} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Higher convolutions of Ramanujan sums

Theorem 0.3 (G., Murty, 2024)

For fixed integers $a_1, \cdots, a_k$ and $q_1, \cdots, q_k$, we have

$$f(q_1, ..., q_k) = \sum_{d_1|q_1, ..., d_k|q_k} d_1 \mu\left(\frac{q_1}{d_1}\right) \cdots d_k \mu\left(\frac{q_k}{d_k}\right) \frac{g(d_1, ..., d_k)}{[d_1, ..., d_k]}.$$

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For fixed integers $a_1, \cdots, a_k$ *and* $q_1, \cdots, q_k$*, we have*

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- Since $g(d_1, ..., d_k)$ is multiplicative, we see that $f(n_1, ..., n_k)$ is multiplicative.

Multiplicative functions of several variables

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- An arithmetical function of several variables is a map $f : \mathbb{N}^k \to \mathbb{C}$. We say f is multiplicative if

$$f(m_1, ..., m_k)f(n_1, ..., n_k) = f(m_1 n_1, ..., m_k n_k)$$

provided $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$.

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- For multiplicative functions f, we have a formal Dirichlet series along with an Euler product:

$$\sum_{\underline{n}=\underline{1}}^{\infty} \frac{f(n_1, \ldots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}} = \prod_p \left(\sum_{v_1, \ldots, v_k = 0}^{\infty} \frac{f(p^{v_1}, \ldots, p^{v_k})}{p^{v_1 s_1} \cdots p^{v_k s_k}} \right).$$

Estimation of $f(p^{v_1}, ..., p^{v_k})$

Lemma 2

For $0 \leq v_i \leq 1$ *for* $1 \leq i \leq k$*, we have*

$$f(p^{v_1}, ..., p^{v_k}) = (-1)^{|S|} + \frac{(-1)^{|S|}}{p} \sum_{C_i} [(1-p)^{|C_i \cap S|} - 1]$$

where $S = \{i : v_i = 1\}$.

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where $S = \{i : v_i = 1\}$.

We define an equivalence relation on $\{1, 2, ..., k\}$ using T. We say $i \sim j$ if and only if $a_i \equiv a_j \pmod{p}$. This partitions T into equivalence classes C_i. Note that $b(p)$ is the number of equivalence classes.

Heuristic proof of k-tuple conjecture

From Lemma 2, we have

$$\sum_{v_1,\dots,v_k \geq 0} \frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})} f(p^{v_1},\dots,p^{v_k})$$

$$= \sum_{v_1,\dots,v_k \geq 0} \frac{\mu(p^{v_1})\cdots\mu(p^{v_k})}{\phi(p^{v_1})\cdots\phi(p^{v_k})} \left\{ (-1)^{|S|} + \frac{(-1)^{|S|}}{p} \sum_{C_i} [(1-p)^{|C_i \cap S|} - 1] \right\}$$

$$= \left(1 - \frac{b(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}.$$

This gives a heuristic proof of k-tuple conjecture.

Thank You!