Filtrations, Mild groups and Arithmetic in an Equivariant context

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Introduction

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- Current proof, Roquette-Wingberg.

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Pro-p groups and presentations

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- Surjection $F \to G$, with kernel R.
- r is the minimal number of generators of R as a closed normal subgroup of F.
- $d = \dim_{\mathbb{F}_p} H^1(G; \mathbb{F}_p) = \dim_{\mathbb{F}_p}(G/G^p[G;G])$ and $r = \dim_{\mathbb{F}_p} H^2(G; \mathbb{F}_p) = \dim_{\mathbb{F}_p}(R/R^p[R;F])$.

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- We have $F_1 = \mathbb{Z}_p$, and $F_n = \widehat{F_{n-1} * \mathbb{Z}_p}$.

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- K a local field, $\chi(\kappa) \neq p$ and $\mu_p \subset K$. Then

$$\mathrm{Gal}(\hat{K}/K) \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_p := \langle \sigma; \tau | \quad \tau^{|\kappa|-1} = [\sigma; \tau] \rangle.$$

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σ is a Frobenius, τ is a generator of the inertia subgroup and exact sequence:

$$1 \to \mathrm{Gal}(\hat{K}/T_K) \to \mathrm{Gal}(\hat{K}/K) \to \mathrm{Gal}(T_K/K) \to 1.$$

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Notations

- $Alp(G) := \varprojlim_N \mathbb{F}_p[G/N]$ is the completed group algebra of G.
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- $G_n := \{g \in G; g-1 \in Alp_n(G)\}$: Zassenhaus filtration of G,

$$\mathrm{Grad}(G) := \bigoplus_{n\in\mathbb{N}} G_n/G_{n+1}, \quad a_n := \dim_{\mathbb{F}_p}(G_n/G_{n+1}).$$

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We can quote [Labute 1985] and [Mináč-Tân 2015], who studied these filtrations for some pro-p groups (free, one relators...).

Example

- If $G := \mathbb{Z}/p\mathbb{Z}$, then $Alp(G) \simeq \mathbb{F}_p[X]/(X^p - 1)$, and:

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- If G is free with d generators, then $Alp(G) \simeq \mathbb{F}_p\langle\langle X_1; \ldots; X_d\rangle\rangle$, and

$$gocha(G, t) := \frac{1}{1 - dt}.$$

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- If $G := \mathbb{Z}_p^d$, then $Alp(G) \simeq \mathbb{F}_p[\![X_1; \ldots; X_d]\!]$, and

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- We can also compute $gocha(G, t)$, when $\mathrm{cd}(G) \leq 2$.

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- Magnus' isomorphism:

$$\phi\colon Alp(F) \simeq \mathbb{F}_p\langle\langle X_j; 1 \leq j \leq d\rangle\rangle$$
$$x_j \mapsto X_j + 1.$$

Working on quotient of Series

- Define E the algebra $\mathbb{F}_p\langle\langle X_j; 1 \leq j \leq d\rangle\rangle$ filtered by $\deg(X_j) = 1$, $\{E_n\}_{n\in\mathbb{N}}$ its filtration.

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- Let n_j be the weight of ρ_j, i.e $\rho_j \in E_{n_j} \setminus E_{n_j+1}$. Define $\overline{\rho_j}$ the image of ρ_j in $E_{n_j}/E_{n_j+1} \subset \mathscr{E}$.

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- Observe $\langle\overline{\rho_j}\rangle \subset \mathscr{I}(R)$. Mild criterion gives equality.
- Define $r(t) := \sum_j t^{n_j}$.
- Result:

$$gocha(G, t)(1 - dt + r(t)) \geq 1.$$

Consequence Golod-Shafarevich criterion

Golod-Shafarevich Theorem

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Corollary

If G is finite, then

$$d^2 < 4r.$$

Known results

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Proposition (Jennings-Lazard Formula, Proposition 3.10 in Appendice A [Lazard 1965])

$$gocha(G,t) = \prod_{n \in \mathbb{N}} P_n(t)^{a_n}, \quad where\ P_n(t) := \left(\frac{1-t^{pn}}{1-t^n}\right). \tag{1}$$

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Let us deduce some consequences of Formula (1):

Consequences of Formula 1

Gocha's alternative, Theorem 3.11 of Appendice A.3 [Lazard 1965]

We have the following alternative:

- Either G is an analytic pro-p group, i.e Lie group over $\mathbb{Q}_p$, so there exists an integer n such that $a_n = 0$ and the sequence $(c_n)_{n\in\mathbb{N}}$ has polynomial growth with n.

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- Or G is not an analytic pro-p group, then for every $n \in \mathbb{N}$, $a_n \neq 0$, and the sequence $(c_n)_{n\in\mathbb{N}}$ does admit an exponential growth with n.

Consequences of Formula 1

In 2016, Mináč, Rogelstad and Tân gave an explicit formula relating a_n and c_n, by introducing:

$$\log(gocha(G,t)) := -\sum_{n\in\mathbb{N}} \frac{(1-gocha(G,t))^n}{n} := \sum_{n\in\mathbb{N}} b_n t^n.$$

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Proposition (Proposition 3.4 of [Mináč, Rogelstad and Tân 2016])

If we write $n = mp^k$, with m coprime to p, then

$$a_n = w_m + w_{mp} + \cdots + w_{mp^k};$$
$$\text{where } w_n := \frac{1}{n}\sum_{m|n} \mu(n/m) m b_m \quad \text{and} \quad \mu \text{ is the Möbius function.} \tag{2}$$

Cohomological dimension

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- $$gocha(G,t) := \frac{1}{1 - dt + r(t)},$$

  implies

  $$cd(G) = 2 \quad \text{and} \quad \dim_{\mathbb{F}_p}(H^2(G, \mathbb{F}_p)) = r(1).$$

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Eigenspaces

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$$M_\chi := \{x \in M; \quad \forall \sigma \in \Delta, \quad \sigma(x) = \chi(\sigma)x\}.$$

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Focus on the graded set $\mathrm{Grad}(G)_\chi := \bigoplus_n (G_n/G_{n+1})_\chi$ and

$$a_n^\chi := \dim_{\mathbb{F}_p}((G_n/G_{n+1})_\chi), \quad c_n^\chi := \dim_{\mathbb{F}_p}((Alp_n(G)/Alp_{n+1}(G))_\chi).$$

New results

Following ideas of [Filip 2011], we introduce:

$$gocha^*(G,t) := \sum_{n\in\mathbb{N}} \left(\sum_{\chi} c_n^{\chi} \chi \right) t^n \in R_{\mathbb{F}_p}[\Delta][[t]].$$

Where $R_{\mathbb{F}_p}[\Delta]$ is the semi-ring generated by χ's over $\mathbb{Z}$.

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Theorem: [H. 2022, Theorem *A*]

$$gocha^*(G,t) = \prod_{n\in\mathbb{N}} \prod_{\chi} P_{n;\chi}(t)^{a_n^{\chi}},$$

where $$P_{n;\chi}(t) := \frac{1-(\chi t^n)^p}{1-\chi t^n}.$$

Consequences

Denominate:

$$\log(gocha^*(G,t)) := -\sum_{n\in\mathbb{N}} \frac{(1-gocha^*(G,t))^n}{n} := \sum_{n\in\mathbb{N}} \left(\sum_{\chi} b_n^{\chi}\chi\right) t^n.$$

Logarithm of series with coefficients in $R_{\mathbb{F}_p}[\Delta] \otimes_{\mathbb{Z}} \mathbb{Q}$ were first studied by [Filip 2011]. We infer:

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Logarithm of series with coefficients in $R_{\mathbb{F}_p}[\Delta] \otimes_{\mathbb{Z}} \mathbb{Q}$ were first studied by [Filip 2011]. We infer:

Proposition: [H. 2022, Formula 2]

Write $n := mp^k$, with m coprime to p, and assume q is coprime with n. Then:

$$a_n^{\chi} = w_m^{\chi} + w_{mp}^{\chi} + \cdots + w_{mp^k}^{\chi},$$

$$\text{where} \quad w_n^{\chi} := \frac{1}{n}\sum_{m|n} \mu(n/m) m b_m^{\chi^{m/n}} \in \mathbb{Q}.$$

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Proposition

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$$\log(1/P) = -\log(P).$$

- *If u is in $tR_{\mathbb{F}_p}[\Delta][[t]]$, then*

$$\log\left(\frac{1}{1-u(t)}\right) = \sum_{\nu=1}^{\infty} \frac{u(t)^{\nu}}{\nu}.$$

Question

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Main Question: For which χ, *is* $\mathrm{Grad}(G)_\chi$ *infinite ?*

Question

Assume G infinite, then Pigeonhole principle: There exists at least one χ such that $\mathrm{Grad}(G)_\chi$ is infinite.

Main Question: For which χ, is $\mathrm{Grad}(G)_\chi$ infinite ?

Partial answer when G is not analytic.

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G is free

Theorem C

Assume that G is a noncommutative free pro-p group.
Then for every χ, the graded set $\mathrm{Grad}(G)_\chi$ is infinite.

G is free

Example

- $\Delta := \langle\sigma\rangle$ of order 2, and χ_0 the unique nontrivial character.
- G is free generated by $\{x_1; \ldots; x_d\}$, and $\sigma(x_i) := x_i^{-1}$.

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- G is free generated by $\{x_1; \ldots; x_d\}$, and $\sigma(x_i) := x_i^{-1}$.
- Observe:

$$gocha^*(G,t) := \frac{1}{1 - d\chi_0 t}, \quad \text{and}$$

$$\log(gocha^*(G,t)) := \sum_n \frac{(d\chi_0)^n}{n} t^n.$$

- Then $c_{2n}^{\mathbb{1}} = d^{2n}, \quad c_{2n+1}^{\mathbb{1}} = 0, \quad c_{2n}^{\chi_0} = 0, \quad c_{2n+1}^{\chi_0} = d^{2n+1}.$
- $b_{2n+1}^{\chi_0} := d^{2n+1}/(2n+1), \quad b_{2n}^{\chi_0} = 0,$
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  $b_{2n+1}^{\chi_0} = 0$, $\quad b_{2n}^{\mathbb{1}} = d^{2n}/(2n)$.
- From [H. 2022, Formula 2], one obtains when $p \neq 3$:

$$a_3^{\chi_0} = w_3^{\chi_0} = \frac{d^3 - d}{3}, \text{ and } a_3^{\mathbb{1}} = 0.$$

$\mathrm{cd}(G) = 2$

Theorem: [H. 2022, Theorem B]

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- Then $\mathrm{cd}(G) = 2$ and

$$gocha(G, t) := 1/(1 - 3t + 2t^2).$$

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- Automorphism σ on G, by:
$\sigma(x) := x^8$, $\sigma(y) := y^{8^2}$ and $\sigma(z) := z^{8^3}$.

Some computations

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$$gocha^*(G,t) := \frac{1}{1 - (\chi_0 + {\chi_0}^2 + {\chi_0}^3)t + ({\chi_0}^3 + {\chi_0}^4)t^2}, \quad \text{and}$$

$$\log(gocha^*(G,t)) = (\chi_0 + {\chi_0}^2 + {\chi_0}^3)t + ({\chi_0}^6/2 + {\chi_0}^5 + {\chi_0}^4/2 + {\chi_0}^2/2)t^2 + ({\chi_0}^9/3 + {\chi_0}^8 + {\chi_0}^7 + {\chi_0}^6/3 + {\chi_0}^3/3)t^3 + \dots.$$

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[H. 2022, Formula 2] gives us:

- $a_2^{\chi_0^5} = 1$, so we conclude that $a_2^{\chi_0^i} = 0$ when $i \neq 5$.
- $a_3^{\chi_0^7} = a_3^{\mathbb{1}} = 1$. Then if $i \notin \{0,7\}$, $a_3^{\chi_0^i} = 0$.

Example

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Then by [H. 2022, Theorem B], for every χ, $\mathrm{Grad}(G)_\chi$ is infinite.

Arithmetic examples: Notations

- Let p be an odd prime.
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- S is tame, i.e for all $\mathfrak{p} \in S$, $N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv 1 \pmod{p}$.
- K_S is the p-maximal extension unramified outside S, and $G_S := \mathrm{Gal}(K_S/K)$.

Koch's computations

Theorem [Koch 2002]

Let $S := \{\mathfrak{p}_i\}$ be a finite tame set of places of a number field K with class number coprime to p, then $G_S := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ admits a presentation with $|S|$ generators and $|S|$ relations.

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$$l_i \equiv \prod_{j \neq i} [x_i; x_j]^{l_{i,j}} \pmod{F_3}.$$

The coefficient $l_{i,j}$ is the linking number of $\mathfrak{p}_i$ and $\mathfrak{p}_j$.

Examples

- Consider [Koch 2002, Example 11.15], take $p = 3$ and $S_0 := \{229, 41\}$. Then the group $G_{S_0} := \mathrm{Gal}(\mathbb{Q}_{S_0}/\mathbb{Q})$ is finite.

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- If we consider $K := \mathbb{Q}(i)$, the primes in S_0 totally split in K. Here $G_S := \mathrm{Gal}(K_S/K)$ admits 4 generators and 4 relations, so G_S is infinite (by GS theorem).

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  In fact, $\mathrm{cd}(G_S) = 2$.

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- Take $p = 3$, and consider $\mathrm{K} := \mathbb{Q}(\sqrt{-163})$.
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- Put $\{p_1 := 31, p_2 := 19, p_3 := 13, p_4 := 337, p_5 := 7, p_6 := 43\}$.
- The class group of K is trivial, the primes p_1, p_2, p_3, p_4, p_5 are inert in K, and the prime p_6 totally splits in K.

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- Define S the primes above the previous set in K, and K_S the maximal p-extension unramified outside S.

FAB example

- Then Δ acts on $G := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$, which is FAB by Class Field Theory.
- We can show that the pro-p group G is mild, so we obtain

$$gocha(\mathbb{F}_p, t) := \frac{1}{1 - 7t + 7t^2}.$$

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- Furthermore:

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- The graded spaces $\mathrm{Grad}(G)_{\mathbb{1}}$ and $\mathrm{Grad}(G)_{\chi_0}$ are both infinite dimensional.
- Moreover, we obtain for instance:

$$a_3^{\chi_0} = 24, \quad \text{and} \quad a_3^{\mathbb{1}} = 39.$$

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