Moments in the Chebotarev Density Theorem

joint work with Régis de la Bretèche and Daniel Fiorilli

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If $(a, q) = 1$, then $\pi(x; q, a) \sim \frac{1}{\phi(q)}\pi(x) \sim \frac{1}{\phi(q)} \frac{x}{\log x}$, as $x \to \infty$.

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$$\psi(x) = \sum_{n \le x} \Lambda(n), \qquad \psi(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n),$$

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PNT in AP's (v2) : $\psi(x; q, a) \sim \frac{1}{\varphi(q)}\psi(x) \sim \frac{1}{\varphi(q)}x$, as $x \to \infty$.

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Theorem (Hooley, 1977)

For $(a, q) = 1$, conditionally on RH and Linear Independence (LI) of the imaginary parts of non negative L-zeros the error term

$$E(x; q, a) := \psi(x; q, a) - (\varphi(q))^{-1} x$$

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is such that for any fixed $r \in \mathbb{N}$,

$$\lim_{q \to \infty} \lim_{X \to \infty} \frac{\varphi(q)^{\frac{r}{2}}}{(\log q)^{\frac{r}{2}}} \frac{1}{\log X} \int_2^X \frac{(E(x; q, a))^r}{x^{\frac{r}{2}}} \frac{\mathrm{d}x}{x} = \mu_r,$$

where

$$\mu_r := \begin{cases} (2n-1) \cdot (2n-3) \cdots 1 & \text{if } r = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

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Question : *uniformity*? A range for q relative to X?

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Ex. $K \geq 1/2 + \delta$. Take $\eta_K(t) = \mathrm{e}^{-K|t|}$.

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$$\mathcal{V}_{s,n} = \frac{1}{U\int_0^\infty \Phi}\int_0^\infty \Phi\Big(\frac{t}{U}\Big)\Big(M_n(\mathrm{e}^t;q,\eta) - m_n(q,\eta)\Big)^s \mathrm{d}t\,.$$

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Here $m_n(q,\eta) = \lim_{T\to\infty}\frac{1}{T}\int_0^T M_n(\mathrm{e}^t;q,\eta)\mathrm{d}t$.

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and $$\psi(x_q; q, a) - \varphi(q)^{-1}\psi(x_q, \chi_{0,q}) \gg \left(\frac{x_q}{\varphi(q)}\right)^{1/2} (\log q)^{1/2}.$$

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Recall Montgomery's conj. : $\psi(x; q, a) - \frac{x}{\varphi(q)} \ll \left(\frac{x}{q}\right)^{1/2} x^{\varepsilon}$ for $q < x$.

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- $\mathrm{Frob}_{\mathfrak{p}} \subset \mathrm{Gal}(L/K)$ the Frobenius conj. class at $\mathfrak{p}$ (lifts to G the Frobenius aut. on the level of residual fields $x \mapsto x^{\mathcal{N}\mathfrak{p}}$).

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Chebotarev Density Theorem

Let $C \subset \mathrm{Gal}(L/K)$ be a conj. class, then

$$\#\{\mathfrak{p} \subset \mathcal{O}_K \text{ unram.} : \mathrm{Frob}_\mathfrak{p} = C, \mathcal{N}\mathfrak{p} \leq x\} \sim_{x\to\infty} \frac{|C|}{|G|}\mathrm{Li}(x),$$

where $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$.

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where $\widehat{t}(\chi) = \frac{1}{|G|} \sum_{g \in G} t(g)\overline{\chi(g)}$, for $\chi \in \mathrm{Irr}(G)$.

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Th. (Bellaïche, 2016)

$K = \mathbb{Q}$, $M = \prod_{p \text{ ram.}} p$, $\lambda_{1,1}(t) = \sum_{\chi \text{ Irr.}} |\widehat{t}(\chi)|\chi(1)$ assuming RH+AC : $\pi(x; L/K, t) - \widehat{t}(1)\mathrm{Li}(x) \ll \lambda_{1,1}(t)\sqrt{x}\log(x|M|G)$

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Chebotarev : $\psi_\eta(x; L/K, t) \sim \widehat{t}(1)\sqrt{x}\mathcal{L}_\eta(\tfrac{1}{2})$, $\left(\mathcal{L}_\eta(u) := \int_{\mathbb{R}} \mathrm{e}^{ux}\eta(x)\mathrm{d}x\right)$.

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Work of Fiorilli–J. implies under RH for ζ_L that $\psi_\eta(x;L/K,t) - \widehat{t}(1)x^{\frac{1}{2}}\mathcal{L}_\eta(\frac{1}{2})$ has average value $\widehat{\eta(0)}z(L/K,t)$.

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Define the n-th moment $\widetilde{M}_n(U, L/K; t, \eta, \Phi)$:

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with $\Phi\colon \mathbb{R}\to\mathbb{R}$ non zero, even, L^1, $\widehat{\Phi}\geq 0$, and $U>0$.

Rmk The convergence of the integral defining $\widetilde{M}_n(U,L/K;t,\eta,\Phi)$ relies on RH + Artin's conj. (AC).

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$$\nu(L/F, t^+; \eta) := \sum_{\chi \in \mathrm{Irr}(G^+)} |\widehat{t^+}(\chi)|^2 b_0(\chi; \widehat{\eta}^2), \ \ b_0(\chi; \widehat{\eta}^2) := \sum_{\rho_\chi \notin \mathbb{R}} \left| \widehat{\eta}\left(\frac{\rho_\chi - \frac{1}{2}}{2\pi i} \right) \right|^2,$$

where ρ_χ runs over the non-trivial zeros of $L(s, L/F, \chi)$.

Finally define

$$w_4(L/F, t^+; \eta) := \frac{\sum_{\chi \in \mathrm{Irr}(G^+)} |\widehat{t^+}(\chi)|^4 b_0(\chi; \widehat{\eta}^2)}{\left(\sum_{\chi \in \mathrm{Irr}(G^+)} |\widehat{t^+}(\chi)|^2 b_0(\chi; \widehat{\eta}^2) \right)^2}.$$

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Main Theorem (de la Bretèche – Fiorilli – J.)

For $m \in \mathbb{N}$, we have the lower bound

$$\widetilde{M}_{2m}(U, L/K; t, \eta, \Phi) \geq \mu_{2m} \nu(L/F, t^+; \eta)^m \Big(1 + O_\eta\Big(m^2 m!\, w_4(L/F, t^+; \eta)\Big)\Big) + O\Big(\frac{(\kappa_\eta [F:\mathbb{Q}]\, \lambda_{1,1}(t^+) \log(\mathrm{rd}_L))^{2m}}{U}\Big),$$

where $\kappa_\eta > 0$ is a constant, and $\mathrm{rd}_L = |\mathrm{disc}(L)|^{1/[L:F]}$.

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Theorem (de la Bretèche – Fiorilli – J.)

Notation/assumptions as in the previous Th.

Assume $S_{t^+} \leq 1 - \kappa_\eta \Big(\log_2(\mathrm{rd}_L + 2) \Big)^{-1}$ where $\kappa_\eta > 0$ is a large enough constant. Then

$$\left| \frac{\nu(L/F, t^+; \eta)}{\alpha(|\widehat{\eta}|^2)[F : \mathbb{Q}] \log(\mathrm{rd}_L) \lambda_{1,2}(t^+)} - 1 \right| \leq S_{t^+} + O_\eta\Big(\frac{1}{\log_2(\mathrm{rd}_L + 2)} \Big),$$

as well as

$$w_4(L/F, t^+; \eta) \ll_{\eta, F} \frac{(\log \log \mathrm{rd}_L)^2}{\log(\mathrm{rd}_L)}.$$

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One has

n	≥ 3	≥ 3	≥ 5
t	$\lvert D_n \rvert \mathbf{1}_e$	$\mathbf{1}_{\{\sigma,\sigma^{-1}\}}$	$2\mathbf{1}_e + \mathbf{1}_{\{\sigma,\sigma^{-1}\}}$
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3. $\mathcal{G} = \left\{ \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \colon c \in \mathbb{F}_p^*,\ d \in \mathbb{F}_p \right\}$, the group of affine transformations of $\mathbb{A}^1_{\mathbb{F}_p}$ has a real irreducible character ϑ of degree $p-1$. One has $S_\vartheta = 1/(p-1)$.

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1. ρ_χ runs through the non-trivial zeros of $L(s, L/K, \chi)$,
2. $A(\chi)$ denotes the (analytic) Artin conductor attached to χ.

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(Recall the def. of $\widetilde{M}_n(U, L/K; t, \eta, \Phi)$:

$$\frac{1}{U\int_0^\infty \Phi}\int_0^\infty \Phi\Big(\frac{u}{U}\Big)\Big(\psi_\eta(\mathrm{e}^u; L/K, t) - \widehat{t}(1)\mathrm{e}^{\frac{u}{2}}\mathcal{L}_\eta(\tfrac{1}{2}) - \widehat{\eta}(0)z(L/K, t)\Big)^n \mathrm{d}u\,.)$$

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Idea : our bounds are best possible in the case $F = \mathbb{Q}$.

Ingredients in the proof of the main result, III

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An approximation of $\widetilde{M}_n(U, L/K; t, \eta, \Phi)$ is given by :

$$\widetilde{D}_n(U, L/F; t, \eta, \Phi) := \frac{(-1)^n}{2\int_0^\infty \Phi} \sum_{\chi_1,\dots,\chi_n \in \mathrm{Irr}(G^+)} \Big(\prod_{j=1}^n \widehat{t}(\chi_j) \Big) \times \sum_{\gamma_{\chi_1},\dots,\gamma_{\chi_n} \neq 0} \widehat{\Phi}\Big(\frac{U}{2\pi}(\gamma_{\chi_1} + \cdots + \gamma_{\chi_n})\Big) \prod_{j=1}^n \widehat{\eta}\Big(\frac{\gamma_{\chi_j}}{2\pi}\Big).$$

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Combinatorics on zeros enables to evaluate $\widetilde{D}_n$ by applying *positivity* to discard contributions possibly violating Linear Independence.

Thanks for your attention !