A Discrete Mean Value of the Riemann Zeta Function and its Derivatives

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Part I: The Riemann Zeta Function
Let \( s = \sigma + it \) with \( \sigma, t \in \mathbb{R} \). The Riemann zeta function is defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (\sigma > 1)
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and Riemann proved that $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1 - s)$$

which gives the meromorphic continuation of $\zeta(s)$ to $\mathbb{C}$ with the simple pole at $s = 1$. 
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By the Euler product above, we see that $\zeta(s) \neq 0$ in the region $\sigma > 1$ and by the functional equation, $\zeta(-2n) = 0$ for all natural numbers $n \geq 1$. 
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A zero of $\zeta(s)$ in the critical strip is called a nontrivial zero and it is denoted by $\rho = \beta + i\gamma$ where $0 \leq \beta \leq 1$ and $\gamma \in \mathbb{R}$. 


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$$N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi e} \right) + O(\log T)$$

as $T \to \infty$. 

But why are we interested in the zeros of $\zeta(s)$? This is simply because we would like to divide by $\zeta(s)$ in order to have a better understanding on the distribution of prime numbers. To make division by $\zeta(s)$ meaningful, we need to know that $\zeta(s) \neq 0$. 
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More precisely, for $\sigma = \Re(s) > 1$, we have

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

which gives

$$-\log \zeta(s) = \sum_p \log \left(1 - \frac{1}{p^s}\right)$$
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and by differentiation with respect to $s$, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{p^{-s} \log p}{1 - \frac{1}{p^s}} = \sum_p p^{-s} \log p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right)$$

$$= \sum_{j \geq 1} \sum_p \frac{\log p}{p^{js}}.$$
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For a natural number $n$, let $\Lambda(n)$ be the von Mangoldt function defined by $\Lambda(n) = \log p$ if $n$ is a prime power $p^j$ for some $j \geq 1$, and $\Lambda(n) = 0$ otherwise. Then

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad (\sigma > 1).$$
Thus the function $-\frac{\zeta'}{\zeta}(s)$ is closely related to prime powers. For $x \geq 2$, by the Residue Theorem, we have

$$\frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \sum_{n=1}^{\infty} \frac{\Lambda(n) x^s}{n^s} \frac{ds}{s} = \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \frac{(\frac{x}{n})^s}{s} ds = \sum_{n \leq x} \Lambda(n) + O(\log x)$$
Thus the function \(-\frac{\zeta'}{\zeta}(s)\) is closely related to prime powers. For \(x \geq 2\), by the Residue Theorem, we have

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\]

and

\[
\frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s} \, ds = x - \sum_{\rho} \frac{x^\rho}{\rho} + O(1)
\]

and thus

\[
\sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho} + O(\log x).
\]
Preliminaries

The Prime Number Theorem, conjectured by Gauss in 1792 and proved by Hadamard and de la Vallée Poussin in 1896 independently, is the statement that

\[ \pi(x) = \left| \{ p \leq x : p \text{ prime} \} \right| \sim \frac{x}{\log x}, \quad (x \to \infty) \]
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or equivalently that

\[ \sum_{n \leq x} \Lambda(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} + O(\log x) = x + o(x), \quad (x \to \infty). \]

Thus the real parts of the nontrivial zeros \( \rho \) of \( \zeta(s) \) play an important role to control the error term in the Prime Number Theorem.
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The best known zero-free region for the Riemann zeta function is due to Korobov and Vinogradov independently in 1958 that \( \zeta(s) \neq 0 \) in the region

\[ \sigma > 1 - \frac{C}{(\log |t|)^{2/3} (\log \log |t|)^{1/3}} \]

for some positive constant \( C \) and \( |t| \geq 3 \) whereas the Riemann Hypothesis is the conjecture that \( \zeta(s) \neq 0 \) if \( \sigma > \frac{1}{2} \).
Part II: Main Results
The discrete $2k^{\text{th}}$ moment of the $m^{\text{th}}$ derivative of the Riemann zeta function is the sum

$$\sum_{\rho \in \mathbb{C}, 0 < \gamma \leq T} \left| \zeta^{(m)}(\rho) \right|^{2k}, \quad (k \in \mathbb{R}, \ m \in \mathbb{N}).$$

One of the reasons to study this object is to have a better understanding on the average size of the derivatives of $\zeta(s)$ at its zeros. Moreover, such moments can produce results on the large or small gaps between the ordinates of the zeros of $\zeta(s)$, and multiplicities of the zeros, and the summatory function of the Möbius function.

Assuming the Riemann Hypothesis, Gonek proved that

$$\sum_{\rho \in \mathbb{C}, 0 < \gamma \leq T} \left| \zeta^{(m)}(\rho) \right|^{2k} \sim T^{2k} \pi \log^4 T,$$

and no other asymptotic formula is known even conditionally.
Discrete Moments of the Riemann Zeta Function

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\sum_{0<\gamma \leq T} \left| \zeta'(\rho) \right|^2 \sim \frac{T}{24\pi} \log^4 T
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and no other asymptotic formula is known even conditionally.
Gonek and Hejhal independently conjectured that

\[ T \left( \log T \right)^{k(k+2)+1} \ll \sum_{0 < \gamma \leq T} \left| \zeta'(\rho) \right|^{2k} \ll T \left( \log T \right)^{k(k+2)+1} \]

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As an upper bound, Kirila proved, under the assumption of the Riemann Hypothesis, that

\[ \sum_{0 < \gamma \leq T} \left| \zeta^{(m)}(\rho) \right|^{2k} \ll T \left( \log T \right)^{k(k+2m)+1} \]

for \( k > 0 \) and \( m \in \mathbb{N} \).
Discrete Moments of the Riemann Zeta Function

Gonek and Hejhal independently conjectured that

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Milinovich and Ng proved under the assumption of the Generalized Riemann Hypothesis that

\[ \sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{2k} \gg T (\log T)^{k(k+2)+1} \]  \hspace{1cm} (1)

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for \( k \in \mathbb{N} \). Very recently, Heap, Li and Zhao obtained the same lower bound in (1) for rational \( k \leq 0 \) assuming the Riemann Hypothesis and the simplicity of the zeros.
Theorem (Kübra Benli, E., Nathan Ng)

Assume the Riemann Hypothesis. Let \( k, m \geq 1 \) be natural numbers. We have

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\sum_{0 < \gamma \leq T} \left| \zeta^{(m)}(\rho) \right|^{2k} \gg T \left( \log T \right)^{k(k+2m)+1}.
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$$\sum_{0 < \gamma \leq T} \left| \zeta^{(m)}(\rho) \right|^{2k} \gg T (\log T)^{k(k+2m)+1}.$$ 

The result above is obtained by a general result on the sum

$$S(\alpha, T, X, Y) := \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha)X(\rho)Y(1 - \rho)$$

where $X(s)$ and $Y(s)$ are some Dirichlet polynomials and the shift $\alpha \in \mathbb{C}$ satisfies $|\alpha| \ll \frac{1}{\log T}$. 
Definitions and Assumptions

Let

\[ X(s) = \sum_{n \leq N} \frac{x(n)}{n^s}, \]
\[ Y(s) = \sum_{n \leq N} \frac{y(n)}{n^s}, \]

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Assume that \( N \ll T^\vartheta \) for some \( 0 < \vartheta < \frac{1}{2} \). Assume further that the submultiplicativity condition

\[ x(mn) \ll |x(m)x(n)| \]
\[ y(mn) \ll |y(m)y(n)| \]

holds for all natural numbers \( m \) and \( n \).
Definitions and Assumptions

Our main result has two parts by using the following assumptions.

**Divisor Bound Assumption:** Assume that there exist $k_1, k_2, \ell_1, \ell_2 \geq 1$ such that

\[
x(n) \ll \tau_{k_1}(n)(\log n)^{\ell_1},
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y(n) \ll \tau_{k_2}(n)(\log n)^{\ell_2}
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where $\tau_k(\cdot)$ is the $k$-fold divisor function given by the coefficients of $\zeta(s)^k$. 

GRH Conjecture: There exists $\Theta \in \left[\frac{1}{2}, 1\right)$ such that for all $q \geq 1$ and for all Dirichlet characters $\chi$ modulo $q$, the Dirichlet $L$-functions $L(s, \chi)$ have no zeros in the region $\sigma > \Theta$. 

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Let $\alpha \in \mathbb{C}$ such that $|\alpha| \leq \frac{1}{15 \log T}$ and define

$$s_\alpha(n) := n^\alpha.$$

For a natural number $k$, define

$$\Phi(s, k) := \prod_{p \mid k} \left(1 - p^{-s}\right), \quad (s \in \mathbb{C}).$$
For $k, h \in \mathbb{N}$, define

$$F_{\alpha, h, k}(T) := \frac{T}{2\pi} \left( \prod_{k=1}^{l} \frac{\zeta'}{\zeta} (1 + \alpha) - \frac{\Lambda(k)}{h^\alpha \Phi(1 + \alpha, k)} - \frac{k}{\varphi(k)} \Phi(\alpha, k) \zeta(1 - \alpha) \left( \frac{T}{2\pi k} \right)^{-\alpha} \right).$$
Main Result

For $k, h \in \mathbb{N}$, define

$$F_{\alpha, h, k}(T) := \frac{T}{2\pi} \left( \frac{1}{h} \sum_{k=1}^{\frac{N}{h}} \frac{\zeta'}{\zeta} (1 + \alpha) - \frac{\Phi(1 + \alpha, k)}{h^{\alpha} \Phi(1 + \alpha, k)} - \frac{k}{\varphi(k)} \Phi(\alpha, k) \zeta(1 - \alpha) \frac{\left( \frac{T}{2\pi k} \right)^{-\alpha}}{1 - \alpha} \right).$$

Theorem (Kübra Benli, E., Nathan Ng)

With the definitions and the assumptions above, we have

$$S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha) X(\rho) Y(1 - \rho)$$

$$= \frac{T}{2\pi} \log \left( \frac{T}{2\pi e} \right) \sum_{n \leq N} \left( s_{-\alpha} \ast x \right)(n) y(n) \frac{1}{n} - \frac{T}{2\pi} \sum_{n \leq N} \left( \Lambda \ast s_{-\alpha} \ast x \right)(n) y(n) \frac{1}{n}$$

$$+ \sum_{g \leq N} \sum_{n \leq N/g} y(g) x(gk) F_{\alpha, h, k}(T) + \tilde{\varepsilon}$$

where the error term $\tilde{\varepsilon}$ satisfies the following bounds.
Theorem (Kübra Benli, E., Nathan Ng)

1. Under the Divisor Bound Assumption, we have

\[ \tilde{\mathcal{E}} \ll T (\log T)^{-A} \]

for any \( A > 0 \).
Main Result

Theorem (Kübra Benli, E., Nathan Ng)

1. **Under the Divisor Bound Assumption**, we have

   \[ \tilde{E} \ll T (\log T)^{-A} \]

   for any \( A > 0 \).

2. **Under the GRH(\( \Theta \)) Conjecture**, we have

   \[ \tilde{E} \ll T^{\Theta + \varepsilon} \left( \left\| \frac{y(n)}{n^\Theta} \right\|_1 \left\| n^{1/2} x(n)(1 \ast |y|)(n) \right\|_1 \right) \]
   
   \[ + T^{\Theta + \varepsilon} \left( \left\| \frac{x(n)y(n)}{n} \right\|_1 \left\| \frac{y(n)}{n^\Theta} \right\|_1 \left\| \frac{x(n)}{n} \right\|_1 \left\| \frac{x(n)}{n^{2-\Theta}} \right\|_1 \right) \]
   
   \[ + T^{\frac{1}{2} + \varepsilon} \left( \left\| x \right\|_1 \left\| \frac{y(n)}{n} \right\|_1 + \left\| y \right\|_1 \left\| \frac{x(n)}{n} \right\|_1 \right) \]

   for any \( \varepsilon > 0 \).
Main Result with Higher Derivatives

For $m \geq 1$, define

$$S_m(T, X, Y) := \frac{d}{d\alpha^m}(S(\alpha, T, X, Y)) \bigg|_{\alpha=0} = \sum_{0<\gamma\leq T} \zeta^{(m)}(\rho)X(\rho)Y(1-\rho).$$

Since the error term in our main result for $S(\alpha, T, X, Y)$ is independent of $\alpha$, we can apply the Cauchy Integral Formula to $S(\alpha, T, X, Y)$ to estimate the $m^{th}$ derivative $S_m(T, X, Y)$. 
Theorem (Kübra Benli, E., Nathan Ng)

For $m \geq 1$, we have

$$S_m(T, X, Y) = \frac{(-1)^{m+1}}{m+1} \frac{T}{2\pi} \sum_{g \leq N} \sum_{h \leq N/g} \frac{y(gh)x(g)}{gh} \left( P_{m+1} \left( \log \left( \frac{T}{2\pi} \right) \right) - Q_{m+1}(\log h) \right)$$

$$+ \frac{T}{2\pi} \sum_{g \leq N} \sum_{h,k \leq N/g} \frac{y(gh)x(gk)}{gkh} \left( (-1)^{m+1} A_m(h, k) + B_m(k, T) \right)$$

$$+ (-1)^m \frac{T}{2\pi} \log \left( \frac{T}{2\pi e} \right) \sum_{n \leq N} \frac{(\log^m * x)(n)y(n)}{n}$$

$$+ (-1)^{m+1} \frac{T}{2\pi} \sum_{n \leq N} \frac{(\Lambda \ast \log^m * x)(n)y(n)}{n} + \tilde{C}$$

where $P_{m+1}$ and $Q_{m+1}$ are monic polynomials of degree $m + 1$ and $A_m(h, k)$ and $B_m(k, T)$ are some arithmetic weights.
Part III: Ideas in the Proofs
Our aim is to estimate the sum

\[ S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha)X(\rho)Y(1 - \rho) \]

where \( T \) is large and \( |\alpha| \leq \frac{1}{15 \log T} \).
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Let \( \kappa := 1 + \frac{1}{\log T} \).
Our aim is to estimate the sum

\[ S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha)X(\rho)Y(1 - \rho) \]

where \( T \) is large and \( |\alpha| \leq \frac{1}{15 \log T} \).

Let \( \kappa := 1 + \frac{1}{\log T} \) and \( \mathcal{C} \) be the positively oriented rectangle with vertices at \( \kappa + i, \kappa + iT, 1 - \kappa + iT \) and \( 1 - \kappa + i \).
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Let \( \kappa := 1 + \frac{1}{\log T} \) and \( \mathcal{C} \) be the positively oriented rectangle with vertices at \( \kappa + i, \kappa + iT, 1 - \kappa + iT \) and \( 1 - \kappa + i \).

By the Residue Theorem, we have
\[ S(\alpha, T, X, Y) = -\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\zeta'}{\zeta} (1 - s)\zeta(s + \alpha)X(s)Y(1 - s) \, ds. \]
Sketch Proof

Our aim is to estimate the sum

\[ S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha)X(\rho)Y(1 - \rho) \]

where \( T \) is large and \(|\alpha| \leq \frac{1}{15 \log T}\).

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By the Residue Theorem, we have

\[ S(\alpha, T, X, Y) = -\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\zeta'(1-s)\zeta(s + \alpha)X(s)Y(1-s)}{\zeta} \, ds. \]

Let \( S_R \) and \( S_L \) denote the integrals over the right vertical line and the left vertical line of the contour \( \mathcal{C} \), respectively.
Sketch Proof

Our aim is to estimate the sum

\[ S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha)X(\rho)Y(1 - \rho) \]

where \( T \) is large and \(|\alpha| \leq \frac{1}{15 \log T}\).

Let \( \kappa := 1 + \frac{1}{\log T} \) and \( \mathcal{C} \) be the positively oriented rectangle with vertices at \( \kappa + i, \kappa + iT, 1 - \kappa + iT \) and \( 1 - \kappa + i \).

By the Residue Theorem, we have

\[ S(\alpha, T, X, Y) = -\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\zeta'}{\zeta} (1 - s)\zeta(s + \alpha)X(s)Y(1 - s) \, ds. \]

Let \( S_R \) and \( S_L \) denote the integrals over the right vertical line and the left vertical line of the contour \( \mathcal{C} \), respectively. Then, by controlling the contributions of the horizontal parts via the convexity bounds, we have

\[ S(\alpha, T, X, Y) = S_R + S_L + O(\varepsilon_1) \]
Sketch Proof

Our aim is to estimate the sum

$$S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha)X(\rho)Y(1 - \rho)$$

where $T$ is large and $|\alpha| \leq \frac{1}{15 \log T}$.

Let $\kappa := 1 + \frac{1}{\log T}$ and $C$ be the positively oriented rectangle with vertices at $\kappa + i, \kappa + iT, 1 - \kappa + iT$ and $1 - \kappa + i$.

By the Residue Theorem, we have

$$S(\alpha, T, X, Y) = -\frac{1}{2\pi i} \int_C \frac{\zeta'}{\zeta} (1 - s)\zeta(s + \alpha)X(s)Y(1 - s) \, ds.$$ 

Let $S_R$ and $S_L$ denote the integrals over the right vertical line and the left vertical line of the contour $C$, respectively. Then, by controlling the contributions of the horizontal parts via the convexity bounds, we have

$$S(\alpha, T, X, Y) = S_R + S_L + O(\epsilon_1)$$

where

$$\epsilon_1 \ll T^{\frac{1}{2} + \epsilon} \left( \|x\|_1 \left\| \frac{y(n)}{n} \right\|_1 + \|y\|_1 \left\| \frac{x(n)}{n} \right\|_1 \right).$$
For the contribution of the right edge

\[ S_R = -\frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} \frac{\zeta'}{\zeta} (1-s)\zeta(s+\alpha)X(s)Y(1-s) \, ds, \]
Sketch Proof

For the contribution of the right edge

\[ S_R = -\frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} \frac{\zeta'(1-s)}{\zeta(s)} \zeta(s + \alpha) X(s) Y(1-s) \, ds, \]

we use

\[ \frac{\zeta'(1-s)}{\zeta} = \frac{\chi'(s)}{\chi} - \frac{\zeta'(s)}{\zeta} = -\log \left( \frac{|t|}{2\pi} \right) - \frac{\zeta'(s)}{\zeta} + O \left( |t|^{-1} \right) \]

where \( \chi(\cdot) \) is the functional equation factor for the Riemann zeta function.
Sketch Proof

For the contribution of the right edge

\[ S_R = -\frac{1}{2\pi i} \int_{\kappa+i}^{\kappa+iT} \frac{\zeta'(1-s)\zeta(s+\alpha)X(s)Y(1-s)}{\zeta} \, ds, \]

we use

\[ \frac{\zeta'(1-s)}{\zeta} = \frac{\chi'(s)}{\chi} - \frac{\zeta'(s)}{\zeta} = -\log \left( \frac{|t|}{2\pi} \right) - \frac{\zeta'(s)}{\zeta} + O \left( |t|^{-1} \right) \]

where \( \chi(\cdot) \) is the functional equation factor for the Riemann zeta function.

Then by using the underlying Dirichlet series of the integrand in \( S_R \), we have

\[ S_R = \frac{T}{2\pi} \log \left( \frac{T}{2\pi e} \right) \sum_{n \leq N} \frac{(s-\alpha \ast x)(n)y(n)}{n} - \frac{T}{2\pi} \sum_{n \leq N} \frac{(\Lambda \ast s-\alpha \ast x)(n)y(n)}{n} + O(e_1). \]
Sketch Proof

For the contribution of the left edge, $S_L$, we use the functional equation

$$\zeta(s + \alpha) = \chi(s + \alpha)\zeta(1 - s - \alpha)$$

to rewrite $S_L$ as

$$S_L = -\frac{1}{2\pi i} \int_{1-\kappa+iT}^{1-\kappa+i} \frac{\zeta'}{\zeta} (1-s) \chi(s + \alpha) \zeta(1-s - \alpha) X(s) Y(1-s) ds$$

$$= \frac{1}{2\pi} \int_1^T \frac{\zeta'}{\zeta} (\kappa + it) \chi(1 - \kappa - it + \alpha) \zeta(\kappa + it - \alpha) X(1 - \kappa - it) Y(\kappa + it) dt$$
Sketch Proof

For the contribution of the left edge, $S_L$, we use the functional equation

$$\zeta(s + \alpha) = \chi(s + \alpha)\zeta(1 - s - \alpha)$$

to rewrite $S_L$ as

$$S_L = -\frac{1}{2\pi i} \int_{1-\kappa+iT}^{1-\kappa+i} \frac{\zeta'(1-s)\chi(s + \alpha)\zeta(1-s - \alpha)X(s)Y(1-s)}{\zeta} ds$$

where we use the notation $X(s) = \sum_{n \leq N} \overline{x(n)}/n^s$ and $Y(s) = \sum_{n \leq N} \overline{y(n)}/n^s$.

$$= \frac{1}{2\pi} \int_{1}^{T} \frac{\zeta'(\kappa + it)\chi(1 - \kappa - it + \overline{\alpha})\zeta(\kappa + it - \overline{\alpha})\overline{X}(1 - \kappa - it)\overline{Y}(\kappa + it)}{\zeta} dt$$
Sketch Proof

For the contribution of the left edge, $S_L$, we use the functional equation
\[ \zeta(s + \alpha) = \chi(s + \alpha)\zeta(1 - s - \alpha) \]
to rewrite $S_L$ as
\[
S_L = -\frac{1}{2\pi i} \int_{1-\kappa+iT}^{1-\kappa+i} \frac{\zeta'}{\zeta} (1-s) \chi(s + \alpha)\zeta(1-s - \alpha) X(s) Y(1-s) \, ds
\]
\[
= \frac{1}{2\pi} \int_{1}^{T} \frac{\zeta'}{\zeta} (\kappa + it) \chi(1 - \kappa - it + \bar{\alpha})\zeta(\kappa + it - \bar{\alpha}) \overline{X}(1 - \kappa - it) \overline{Y}(\kappa + it) \, dt
\]
where we use the notation $\overline{X}(s) = \sum_{n \leq N} \overline{x(n)}/n^s$ and $\overline{Y}(s) = \sum_{n \leq N} \overline{y(n)}/n^s$. Define
\[
S_L(\gamma) := \frac{1}{2\pi} \int_{1}^{T} \frac{\zeta(\kappa + \gamma + it)}{\zeta(\kappa + it)} \chi(1 - \kappa - it + \bar{\alpha})\zeta(\kappa + it - \bar{\alpha}) \overline{X}(1 - \kappa - it) \overline{Y}(\kappa + it) \, dt
\]
for $\gamma \in \mathbb{C}$ with $|\gamma| \leq \frac{1}{15 \log T}$. 

Sketch Proof

For the contribution of the left edge, $S_L$, we use the functional equation
\[ \zeta(s + \alpha) = \chi(s + \alpha)\zeta(1 - s - \alpha) \]
to rewrite $S_L$ as
\[
S_L = -\frac{1}{2\pi i} \int_{1-\kappa+iT}^{1-\kappa+i} \frac{\zeta'}{\zeta} (1-s)\chi(s+\alpha)\zeta(1-s-\alpha)X(s)Y(1-s) \, ds
\]

\[
= \frac{1}{2\pi} \int_1^T \frac{\zeta'}{\zeta} (\kappa + it)\chi(1-\kappa - it + \overline{\alpha})\zeta(\kappa + it - \overline{\alpha})\overline{X}(1-\kappa - it)\overline{Y}(\kappa + it) \, dt
\]

where we use the notation $\overline{X}(s) = \sum_{n \leq N} \overline{x(n)}/n^s$ and $\overline{Y}(s) = \sum_{n \leq N} \overline{y(n)}/n^s$. Define
\[
S_L(\gamma) := \frac{1}{2\pi} \int_1^T \frac{\zeta(\kappa + \gamma + it)}{\zeta(\kappa + it)} \chi(1-\kappa - it + \overline{\alpha})\zeta(\kappa + it - \overline{\alpha})\overline{X}(1-\kappa - it)\overline{Y}(\kappa + it) \, dt
\]
for $\gamma \in \mathbb{C}$ with $|\gamma| \leq \frac{1}{15 \log T}$. Then we have
\[
S_L = \frac{d}{d\gamma} S_L(\gamma) \bigg|_{\gamma=0}.
\]
Now our aim is to estimate the integral

\[
S_L(\gamma) = \frac{1}{2\pi} \int_1^T \frac{\zeta(\kappa + \gamma + it)}{\zeta(\kappa + it)} \chi(1 - \kappa - it + \bar{\alpha})\zeta(\kappa + it - \bar{\alpha})X(1 - \kappa - it)Y(\kappa + it) \, dt.
\]
Sketch Proof

Now our aim is to estimate the integral

\[ S_L(\gamma) = \frac{1}{2\pi} \int_1^T \frac{\zeta(\kappa + \gamma + it)}{\zeta(\kappa + it)} \chi(1 - \kappa - it + \bar{\alpha})\zeta(\kappa + it - \bar{\alpha})X(1 - \kappa - it)Y(\kappa + it) dt. \]

For \( \Re(w) = \kappa - \Re(\bar{\alpha}) > 1 \), define

\[ A(w) := \frac{\zeta(w + \bar{\alpha} + \gamma)}{\zeta(w + \bar{\alpha})} \zeta(w)Y(w + \bar{\alpha}) = \sum_{m=1}^{\infty} \frac{\sum_{m_1m_2m_3m_4=m} \mu(m_1)m_1^{-\bar{\alpha}}m_2^{-\bar{\alpha}}-\gamma y(m_4)m_4^{-\bar{\alpha}}}{m_4 \leq N} m^w \]

\[ = \sum_{m=1}^{\infty} \frac{a(m)}{m^w} \]
Sketch Proof

Now our aim is to estimate the integral

\[ S_L(\gamma) = \frac{1}{2\pi i} \int_{\kappa+iT - \alpha}^{\kappa+iT - \alpha} \chi(1 - w)B(1 - w)A(w)\, dw. \]

For \( \Re(w) = \kappa - \Re(\alpha) > 1 \), define

\[ A(w) := \frac{\zeta(w + \alpha + \gamma)}{\zeta(w + \alpha)} \zeta(w) \overline{Y}(w + \alpha) = \sum_{m=1}^{\infty} \frac{\sum_{m_1m_2m_3m_4=m} \mu(m_1)m_1^{-\alpha}m_2^{-\alpha}m_4^{-\alpha} \gamma(m_4)m_4^{-\alpha}}{\sum_{m_1m_2m_3m_4=m} \mu(m_1)m_1^{-\alpha}m_2^{-\alpha}m_4^{-\alpha}} \frac{1}{m^w} = \sum_{m=1}^{\infty} \frac{a(m)}{m^w} \]

and

\[ B(1 - w) := \overline{X}(1 - w - \alpha) = \sum_{k \leq N} \frac{\chi(k)k^\alpha}{k^{1-w}} = \sum_{k \leq N} \frac{b(k)}{k^{1-w}}. \]

Then we have

\[ S_L(\gamma) = \frac{1}{2\pi i} \int_{\kappa+iT - \alpha}^{\kappa+iT - \alpha} \chi(1 - w)B(1 - w)A(w)\, dw. \]
By using the stationary phase method, we have

\[ S_L(\gamma) = \sum_{k \leq N} \frac{b(k)}{k} \sum_{m \leq kT/2\pi} a(m) e(-m/k) + O(e_1) \]

where \( e(-m/k) = e^{-2\pi im/k} \).
By using the stationary phase method, we have

\[ S_L(\gamma) = \sum_{k \leq N} \frac{b(k)}{k} \sum_{m \leq kT/2\pi} a(m) e \left( -\frac{m}{k} \right) + O(\epsilon_1) \]

where \( e \left( -\frac{m}{k} \right) = e^{-2\pi i m/k} \). For the inner sum above, we use the identity

\[ e \left( -\frac{m}{k} \right) = \frac{\mu(k/(k,m))}{\phi(k/(k,m))} + \sum_{q | k} \sum_{\psi \pmod{q} \atop q > 1} \tau(\psi) \sum_{d | m} \psi \left( \frac{m}{d} \right) \delta(q, k, d, \psi) \]

where

\[ \delta(q, k, d, \psi) = \sum_{e | d} \mu(d/e) \phi(k/e) \psi \left( -\frac{k}{eq} \right) \psi \left( \frac{d}{e} \right) \mu \left( \frac{k}{eq} \right) \mu \left( \frac{k}{eq} \right) \]
Sketch Proof

Let

\[ M(\gamma) := \sum_{k \leq N} \frac{b(k)}{k} \sum_{m \leq kT/2\pi} a(m) \frac{\mu(k/(m,k))}{\phi(k/(m,k))} \]
Sketch Proof

Let

\[ M(\gamma) := \sum_{k \leq N} \frac{b(k)}{k} \sum_{m \leq kT/2\pi} a(m) \frac{\mu(k/(m, k))}{\phi(k/(m, k))} \]

and

\[ E(\gamma) := \sum_{k \leq N} \frac{b(k)}{k} \sum_{m \leq kT/2\pi} a(m) \sum_{q | k, q > 1} \sum_{\psi \pmod{q}} \tau(\psi) \sum_{d | m, d | k} \psi \left( \frac{m}{d} \right) \delta(q, k, d, \psi). \]
Let
\[ M(\gamma) := \sum_{k \leq N} \frac{b(k)}{k} \sum_{m \leq kT/2\pi} a(m) \frac{\mu(k/(m,k))}{\phi(k/(m,k))} \]
and
\[ E(\gamma) := \sum_{k \leq N} \frac{b(k)}{k} \sum_{m \leq kT/2\pi} a(m) \sum_{q|k} \tau(\psi) \sum_{\psi \pmod{q} \psi > 1} \sum_{d|m} \psi \left( \frac{m}{d} \right) \delta(q, k, d, \psi). \]
Thus we have
\[ S_L(\gamma) = M(\gamma) + E(\gamma) + O(e_1) \]
and by differentiating with respect to \( \gamma \) via the Cauchy Integral Formula, we have
\[ S_L = M + E + O(e_1). \]
Now we state our results for the terms \( M \) and \( E \).
The term $\mathcal{M}$

**Proposition**

We have

$$\mathcal{M} = \sum_{g \leq N} \sum_{h,k \leq N/g} \frac{y(gh)x(gk)}{ghk} \mathcal{F}_{\alpha,h,k}(T) + O(\epsilon_2)$$

where

$$\mathcal{F}_{\alpha,h,k}(T) = \frac{T}{2\pi} \left( \prod_{k=1}^{\lfloor \frac{T}{2\pi k} \rfloor} \frac{\zeta'}{\zeta} (1 + \alpha) - \frac{\Lambda(k)}{h^\alpha \Phi(1 + \alpha, k)} - \frac{k}{\varphi(k)} \Phi(\alpha, k)\zeta(1 - \alpha) \left( \frac{T}{2\pi k} \right)^{-\alpha} \right)$$

and

$$\epsilon_2 := \begin{cases} T \exp \left( -c\sqrt{\log T} \right) & \text{on the Divisor Bound Assumption,} \\ T^{\Theta + \epsilon} & \text{on the GRH}(\Theta) \text{ Conjecture} \end{cases}$$

for some positive constant $c$. 


The term $\mathcal{M}$

**Proposition**

We have

$$\mathcal{M} = \sum_{g \leq N} \sum_{h, k \leq N/g, (h, k) = 1} \frac{y(gh)x(gk)}{ghk} F_{\alpha, h, k}(T) + O(\varepsilon_2)$$

where

$$F_{\alpha, h, k}(T) = \frac{T}{2\pi} \left( \prod_{k=1}^{\infty} \frac{\zeta'}{\zeta} (1 + \alpha) - \frac{\Lambda(k)}{h^{\alpha} \Phi(1 + \alpha, k)} - \frac{k}{\varphi(k)} \Phi(\alpha, k) \zeta(1 - \alpha) \left( \frac{T}{2\pi k} \right)^{-\alpha} \right)$$

and

$$\varepsilon_2 := \begin{cases} T \exp \left( -c \sqrt{\log T} \right) \\ T^{\Theta + \varepsilon} \left\| \frac{x(n)y(n)}{n} \right\|_1 \left\| \frac{y(n)}{n^{\Theta}} \right\|_1 \left\| \frac{x(n)}{n} \right\|_1 \left\| \frac{x(n)}{n^{2-\Theta}} \right\|_1 \\ \end{cases}$$

on the Divisor Bound Assumption, on the GRH($\Theta$) Conjecture

for some positive constant $c$.

The proof of this result uses a decomposition lemma by Conrey, Ghosh and Gonek to find the generating series for the coefficients appearing in $\mathcal{M}(\gamma)$ and then we estimate the corresponding summatory function by Perron’s formula.
Proposition

We have

\[ \mathcal{E} \ll \begin{cases} T(\log T)^{-A} & \text{on the Divisor Bound Assumption}, \\ T^{\Theta+\epsilon} \left\| \frac{y(n)}{n^\Theta} \right\|_1 \left\| n^{1/2} x(n)(1 * |y|)(n) \right\|_1 & \text{on the GRH}(\Theta) \text{ Conjecture.} \end{cases} \]
The term $\mathcal{E}$

**Proposition**

We have

$$\mathcal{E} \ll \begin{cases} T(\log T)^{-A} & \text{on the Divisor Bound Assumption,} \\ T^{\Theta+\varepsilon} \left\| \frac{y(n)}{n^\Theta} \right\|_1 \left\| n^{1/2} \chi(n)(1 \ast |y|)(n) \right\|_1 & \text{on the GRH}(\Theta) \text{ Conjecture.} \end{cases}$$

The proof of this result uses again the underlying generating series and Perron’s formula in the conjectural case. But in the case where we assume the Divisor Bound Assumption only, we use Heath-Brown’s combinatorial decomposition of $\mu(n)$ and the Large Sieve Inequality as utilized in the work of Heap, Li and Zhao.
The term $\mathcal{E}$

**Proposition**

We have

$$\mathcal{E} \ll \begin{cases} T (\log T)^{-A} & \text{on the Divisor Bound Assumption,} \\ T^{\Theta+\varepsilon} \left\| \frac{y(n)}{n^{\Theta}} \right\|_1 & \text{on the GRH}(\Theta) \text{ Conjecture.} \end{cases}$$

The proof of this result uses again the underlying generating series and Perron’s formula in the conjectural case. But in the case where we assume the Divisor Bound Assumption only, we use Heath-Brown’s combinatorial decomposition of $\mu(n)$ and the Large Sieve Inequality as utilized in the work of Heap, Li and Zhao.

By combining the estimates above, we obtain our main result on the discrete mean value

$$S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha) X(\rho) Y (1 - \rho).$$
Recall: The Main Result

For $k, h \in \mathbb{N}$, recall that

$$F_{\alpha, h, k}(T) = \frac{T}{2\pi} \left( \prod_{k=1}^{h} \frac{\zeta'}{\zeta} (1 + \alpha) - \frac{\Lambda(k)}{h^{\alpha} \Phi(1 + \alpha, k)} - \frac{k}{\varphi(k) \Phi(\alpha, k)} \zeta(1 - \alpha) \left( \frac{T}{2\pi k} \right)^{-\alpha} \right).$$
Recall: The Main Result

For $k, h \in \mathbb{N}$, recall that

$$F_{\alpha, h, k}(T) = \frac{T}{2\pi} \left( \frac{1}{h^\alpha} \sum_{k=1}^{\infty} \frac{\zeta'}{\zeta}(1 + \alpha) - \frac{\Lambda(k)}{h^\alpha \Phi(1 + \alpha, k)} - \frac{k}{\varphi(k)} \Phi(\alpha, k) \zeta(1 - \alpha) \left( \frac{T}{2\pi k} \right)^{-\alpha} \right).$$

Theorem (Kübra Benli, E., Nathan Ng)

With the definitions and the assumptions above, we have

$$S(\alpha, T, X, Y) = \sum_{0 < \gamma \leq T} \zeta(\rho + \alpha) X(\rho) Y(1 - \rho)$$

$$= \frac{T}{2\pi} \log \left( \frac{T}{2\pi e} \right) \sum_{n \leq N} \frac{(s-\alpha * x)(n)y(n)}{n} - \frac{T}{2\pi} \sum_{n \leq N} \frac{(\Lambda * s-\alpha * x)(n)y(n)}{n}$$

$$+ \sum_{g \leq N} \sum_{h, k \leq N/g, (h, k) = 1} \frac{y(gh)x(gk)}{gkh} F_{\alpha, h, k}(T) + \tilde{E}.$$
Recall: Main Result with Higher Derivatives

By using the Cauchy Integral Formula and the previous result, we obtain the following estimate for higher derivatives.

**Theorem (Kübra Benli, E., Nathan Ng)**

For \( m \geq 1 \), we have

\[
S_m(T, X, Y) = \frac{(-1)^{m+1}}{m+1} \frac{T}{2\pi} \sum_{g \leq N} \sum_{h \leq N/g} \frac{y(gh) x(g)}{gh} \left( P_{m+1} \left( \log \left( \frac{T}{2\pi} \right) \right) - Q_{m+1}(\log h) \right) \\
+ \frac{T}{2\pi} \sum_{g \leq N} \sum_{h, k \leq N/g} \frac{y(gh) x(gk)}{gkh} \left( (-1)^{m+1} A_m(h, k) + B_m(k, T) \right) \\
+ (-1)^m \frac{T}{2\pi} \log \left( \frac{T}{2\pi e} \right) \sum_{n \leq N} \frac{(\log^m * x)(n) y(n)}{n} \\
+ (-1)^{m+1} \frac{T}{2\pi} \sum_{n \leq N} \frac{(\Lambda * \log^m * x)(n) y(n)}{n} + \tilde{E}
\]

where \( P_{m+1} \) and \( Q_{m+1} \) are monic polynomials of degree \( m + 1 \) and \( A_m(h, k) \) and \( B_m(k, T) \) are some arithmetic weights.
Sketch Proof of the Corollary

Now our aim to obtain the lower bound

$$\sum_{0 < \gamma \leq T} |\zeta^{(m)}(\rho)|^{2k} \gg T(\log T)^{k^2 + 2km + 1}. $$

for $k, m \in \mathbb{N}$ under the assumption of the Riemann Hypothesis.
Sketch Proof of the Corollary

Now our aim to obtain the lower bound

$$\sum_{0 < \gamma \leq T} |\zeta^{(m)}(\rho)|^{2k} \gg T(\log T)^{k^2+2km+1}.$$  

for $k, m \in \mathbb{N}$ under the assumption of the Riemann Hypothesis. For $0 < \vartheta < \frac{1}{2}$ and sufficiently large $T$, let $N = \xi^k = T^{\vartheta}$. Define

$$C_\xi(s) := \sum_{n \leq \xi} \frac{1}{n^s}.$$
Sketch Proof of the Corollary

Now our aim to obtain the lower bound

$$\sum_{0 \leq \gamma \leq T} |\zeta^{(m)}(\rho)|^{2k} \gg T \left( \log T \right)^{k^2 + 2km + 1}.$$  

for $k, m \in \mathbb{N}$ under the assumption of the Riemann Hypothesis. For $0 < \vartheta < \frac{1}{2}$ and sufficiently large $T$, let $N = \xi^k = T^{\vartheta}$. Define

$$C_{\xi}(s) := \sum_{n \leq \xi} \frac{1}{n^s}.$$  

Observe that

$$\Sigma_1 := \sum_{0 < \gamma \leq T} \zeta^{(m)}(\rho) C_{\xi}(\rho)^{k-1} \overline{C_{\xi}(\rho)}^{k} = \sum_{0 < \gamma \leq T} \zeta^{(m)}(\rho) C_{\xi}(\rho)^{k-1} C_{\xi}(1 - \rho)^k$$

by the assumption of the Riemann Hypothesis.
Sketch Proof of the Corollary

Now our aim to obtain the lower bound

\[
\sum_{0 < \gamma \leq T} |\zeta^{(m)}(\rho)|^{2k} \gg T(\log T)^{k^2+2km+1}.
\]

for \( k, m \in \mathbb{N} \) under the assumption of the Riemann Hypothesis. For \( 0 < \vartheta < \frac{1}{2} \) and sufficiently large \( T \), let \( N = \xi^k = T^\vartheta \). Define

\[
C_\xi(s) := \sum_{n \leq \xi} \frac{1}{n^s}.
\]

Observe that

\[
\Sigma_1 := \sum_{0 < \gamma \leq T} \zeta^{(m)}(\rho)C_\xi(\rho)^{k-1} \overline{C_\xi(\rho)}^k = \sum_{0 < \gamma \leq T} \zeta^{(m)}(\rho)C_\xi(\rho)^{k-1}C_\xi(1 - \rho)^k
\]

by the assumption of the Riemann Hypothesis. By Hölder’s inequality, we have

\[
|\Sigma_1| \leq \left( \sum_{0 < \gamma \leq T} |\zeta^{(m)}(\rho)|^{2k} \right)^{1 \over 2k} \left( \sum_{0 < \gamma \leq T} \left( |C_\xi(\rho)|^{2k-1} \right)^{2k \over 2k-1} \right)^{2k-1 \over 2k}.
\]
Thus, by taking the $2k^{\text{th}}$ power of both sides, we have

$$|\Sigma_1|^{2k} \leq \left( \sum_{0<\gamma<T} |\zeta^{(m)}(\rho)|^{2k} \right) \Sigma_2^{2k-1}$$

where

$$\Sigma_2 := \sum_{0<\gamma\leq T} |C_\xi(\rho)|^{2k}.$$
Sketch Proof of the Corollary

Thus, by taking the $2k^{th}$ power of both sides, we have

$$|\Sigma_1|^{2k} \leq \left( \sum_{0 < \gamma < T} \left| \zeta^{(m)}(\rho) \right|^{2k} \right)^{2k-1}$$

where

$$\Sigma_2 := \sum_{0 < \gamma \leq T} |C_\xi(\rho)|^{2k}.$$

This gives the lower bound

$$\sum_{0 < \gamma \leq T} \left| \zeta^{(m)}(\rho) \right|^{2k} \geq \frac{|\Sigma_1|^{2k}}{|\Sigma_2|^{2k-1}}.$$
Sketch Proof of the Corollary

By using our main result concerning higher derivatives, we have

\[ \Sigma_1 \gg T (\log T)^{k^2+m+1}. \]

By a result of Milinovich and Ng, we have

\[ \Sigma_2 \ll T (\log T)^{k^2+1}. \]

Hence

\[
\sum_{0 < \gamma < T} \left| \zeta^{(m)}(\rho) \right|^{2k} \geq \frac{|\Sigma_1|^{2k}}{\Sigma_2^{2k-1}} \gg \frac{T^{2k} (\log T)^{2k(k^2+m+1)}}{T^{2k-1} (\log T)^{(2k-1)(k^2+1)}} = T (\log T)^{k^2+2km+1}.
\]
THANK YOU!


