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(X, d) a σ-compact metric space, $\mathcal{B}$ its Borel σ-algebra. All measures will be Borel probability measures on σ-compact metric spaces.

*Weak-** will mean the weak-* topology on Borel measures of total variation at most 1. This is a subset of the dual space of $(C_0(X), \|\cdot\|_{\text{sup}})$. It is compact and metrizable.

Mixing for $\mathbb{Z}$-actions

Definition

Let $(X, \mathcal{B}, \mu, T)$ be a Borel probability measure preserving system. We say that it is mixing if for every $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$$

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- U_T^n converges in the weak operator topology to integration against constant functions
- The sequence of measures $(id \times T^n)_*\mu$ converge in the weak-* topology to $\mu \otimes \mu$.

Definition

We say $(X, \mathcal{B}, \mu, T)$ *is mixing of order* k *if for every* $A_1, ..., A_k \in \mathcal{B}$

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Question

(Rokhlin) Does 2-mixing imply 3-mixing?

Partial progress

1. True for *Rank 1 systems* (Kalikow)
2. True for *finite rank systems* (Ryzhikov)
3. True for systems with singular (with respect to Lebesgue) spectral type (Host)
4. Follows from the Hopf argument (Coudène-Hasselblatt-Troubetzkoy).

Mixing for group actions

Let G be a completely metrizable topological group and for each $g \in G$ let T_g be a measure preserving map of (X, μ). Further assume $T_{g_1} T_{g_2} = T_{g_1 g_2}$.
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We say $(X, \mathcal{B}, \mu, G)$ is mixing of order k if for every $A_1, ..., A_k \in \mathcal{B}$ we have

$$\lim_{g_i g_j^{-1} \to \infty} \mu(g_1 A_1 \cap ... \cap g_k A_k) = \mu(A_1)...\mu(A_k).$$

Theorem

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Theorem

(Mozes) When $G = SL(2, \mathbb{R})$, 2-mixing implies mixing of all orders and in particular 3-mixing.

Mozes proved this result in much larger generality. We present this special case for concreteness

Idea of proof that 2-mixing implies 3-mixing

(1) It suffice to show that if $\vec{g}_n = (id, \alpha_n, \beta_n) \in G^3$ is a sequence so that $\alpha_n, \beta_n, \alpha_n^{-1}\beta_n \to \infty$ and $(\vec{g}_n)_*\mu$ weak-*converges to a measure σ then $\sigma = \mu \otimes \mu \otimes \mu$.

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(2) Let (Y, ν), (Z, η) be probability measure spaces, and τ be coupling of them. If (Z, η, T) is ergodic and τ is $(id \times T)$-invariant then $\tau = \nu \otimes \eta$.

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(3) Let σ be as in 1). Either σ is (id, ϕ, ψ)-invariant where $T : X \times X$ by $T(x, y) = (\phi x, \psi y)$ is $\mu \otimes \mu$-ergodic OR σ is (id, id, ψ) invariant where $T = \psi$ acts ergodically on (X, μ).

Proof of theorem

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Justification of (1)

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- G is mixing iff this is automatically $\mu \otimes \mu \otimes \mu$.

Justification of (2)

Proposition

Let (Y, ν), (Z, η) be probability measure spaces, and τ be coupling of them. If (Z, η, T) is ergodic and τ is $(id \times T)$-invariant then $\tau = \nu \times \eta$.

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By *disintegration of measures* applied to projection onto Y, there are probability measure τ_y so that $\tau_y(\{y\} \times Z) = 1$ and $\int_Y \tau_y d\nu = \tau$.

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By *disintegration of measures* applied to projection onto Y, there are probability measure τ_y so that $\tau_y(\{y\} \times Z) = 1$ and $\int_Y \tau_y d\nu = \tau$.
We may identify τ_y with measures $\tilde{\tau}_y$ on Z and by assumption these are T-invariant.

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Thus $\tilde{\tau}_y = \eta$ for ν-almost every y.

Finally, $\tau = \int_Y \tau_y d\nu = \int_Y (\delta_y \otimes \eta) d\nu = \nu \otimes \eta$.

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Let $f \in C_c(X^3)$

$$\int_{X^3} fd((\vec{g}_n)_*\mu) = \int_X f(x, \alpha_n x, \beta_n x) d\mu \qquad (1)$$

$$= \int_X f(hx, \alpha_n hx, \beta_n hx) d\mu. \qquad (2)$$

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Observe that $(h, \alpha_n h \alpha_n^{-1}, \beta_n h \beta_n^{-1})(id, \alpha_n, \beta_n) = (hx, h\alpha_n x, h\beta_n x)$. This gives invariance: $(h, \alpha_n h \alpha_n^{-1}, \beta_n h \beta_n^{-1})$ changes one description of $(g_n)_*\mu$ to another.

Justification of (3) invariance in the limit

Lemma

If $(h_n, \alpha_n h_n \alpha_n^{-1}, \beta_n h_n \beta_n^{-1})$ *converges to* (θ, ϕ, ψ) *then* σ *is* (θ, ϕ, ψ)*-invariant.*

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Let $f \in C_c(X^3)$ and $F(x, y, z) = f(\theta x, \phi y, \psi z) \in C_c(X^3)$.

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$$\begin{aligned}
\int_{X^3} f d\sigma &= \lim_{n\to\infty} \int_X f(x, \alpha_n x, \beta_n x) d\mu && (3) \\
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&= \int_{X^3} F d\sigma && (5)
\end{aligned}$$

Looking for a limit A

Proposition

Assume that whenever $g_n \in SL(2,\mathbb{R})$ goes to infinity we have that for any neighborhood of id, U and bounded set B,

$$g_n U g_n^{-1} \not\subset B$$

for all large n,

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for all large n, then there is a sequence $h_n \in SL(2,\mathbb{R}$ so that

1. $h_n \to id$
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The proposition says that after passing to a subsequence, we may assume σ is (θ, ϕ, ψ)-invariant with $\theta = id$ and at least one of ϕ, ψ not equal to the identity.

Proof of Proposition

Let $\Phi_n : SL(2,\mathbb{R}) \to [1,\infty)$ by

$$\Phi_n(h) = \max\{\|\alpha_n h \alpha_n^{-1}\|, \|\beta_n h \beta_n\|^{-1}\}.$$

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- So for all large n we may choose $h_n \in U$ so that $\max\{\|\alpha h_n \alpha_n^{-1}\|, \|\beta_n h_n \beta_n\|^{-1}\} = 2$.

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- Choosing shrinking U we may assume $h_n \to id$.

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- Because eigenvalues change continuously the eigenvalues of ϕ and ψ are 1.

- Because G is mixing, any element of G that generates an unbounded subgroup is mixing.

Looking for a limit B

Proposition

Whenever $g_n \in SL(2,\mathbb{R})$ goes to infinity we have that for any neighborhood of id, U and bounded set B,

$$g_n U g_n^{-1} \not\subset B$$

for all large n.

This is a computation.

Doing the computation

Let $g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$.

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$$\begin{pmatrix} a_n d_n - a_n c_n s - b_n c_n & -a_n b_n + a_n^2 s + a_n b_n \\ c_n d_n - c_n^2 s - c_n d_n & -b_n c_n + a_n c_n s + a_n d_n \end{pmatrix} =$$

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$$\begin{pmatrix} 1 - a_n c_n s & a_n^2 s \\ -c_n^2 s & 1 + a_n c_n s \end{pmatrix}.$$

For this to be bounded for all small s we need that a_n and c_n are bounded (in n).

Doing the computation II

Similarly, $g_n \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} g_n^{-1} =$

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For this to be bounded for all small s we need that b_n and d_n are also bounded.

This contradicts that g_n is unbounded.

Recap of (3)

So σ is (id, ϕ, ψ)-invariant with at least one of $\phi, \psi \neq id$.

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So σ is $(\mathit{id}, \phi, \psi)$-invariant with at least one of $\phi, \psi \neq \mathit{id}$. If only one is non-identity, it is mixing and thus ergodic and so we have one option for (3).

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Recap of (3)

So σ is (id, ϕ, ψ)-invariant with at least one of $\phi, \psi \neq id$. If only one is non-identity, it is mixing and thus ergodic and so we have one option for (3).

Otherwise they are both mixing, so their product is ergodic and we have the other option for (3).