Sums of Fibonacci numbers close to a power of 2

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Introduction

Definition(Fibonacci and Lucas numbers)

The Fibonacci sequence $(F_n)_{n\geq 0}$ is the binary recurrence sequence defined by $F_0 = 0$, $F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

The sequence of Lucas numbers $(L_n)_{n\geq 0}$ is similarly defined as $L_0 = 2$, $L_1 = 1$, and

$$L_{n+2} = L_{n+1} + L_n \text{ for all } n \geq 0.$$

The Lucas numbers are related to the Fibonacci numbers by the following identity

$$L_k = F_{k-1} + F_{k+1} \text{ for all } k \geq 1.$$

Diophantine Equations

- Bugeaud, Mignotte and Siksek (2006): The only perfect powers among the Fibonacci numbers are 0, 1, 8 and 144 and the only perfect powers in the Lucas sequence are 1 and 4.
- Bravo and Luca (2015): The only solutions $(n, m, a) \in \mathbb{N}^3$ of the Diophantine equation $F_n + F_m = 2^a$ with $n > m > 0$ are

$$(n, m, a) = (2, 1, 1), (4, 1, 2), (4, 2, 2), (5, 4, 3), (7, 4, 4)$$

- Bravo and Bravo (2015): All solutions $(n, m, l, a) \in \mathbb{N}^4$ of the Diophantine equation $F_n + F_m + F_l = 2^a$ with $n > m > l > 0$ are

$$\begin{aligned}(n, m, l, a) = {} & (3, 2, 1, 2), (5, 3, 1, 3), (5, 3, 2, 3), (6, 5, 4, 4), (7, 3, 1, 4), \\ & (7, 3, 2, 4), (8, 6, 4, 5), (10, 6, 1, 6), (10, 6, 2, 6), (11, 9, 5, 7), \\ & (13, 8, 3, 8), (16, 9, 4, 10)\end{aligned}$$

- Ziegler (2022): Let $y > 1$ be a fixed integer, then there exists at most one solution $(n, m, a) \in \mathbb{N}^3$ to the Diophantine equation

$$F_n + F_m = y^a, \ \ n > m > 1, \ \ a > 0,$$

unless $y = 2, 3, 4, 6, 10$. In the case that $y = 2, 3, 4, 6$ or 10 all solutions are listed below:

$$\begin{aligned} y = 2 &: (n, m, a) = (4, 2, 2), (5, 4, 3), (7, 4, 4); \\ y = 3 &: (n, m, a) = (3, 2, 1), (6.2, 2); \\ y = 4 &: (n, m, a) = (4, 2, 1), (7, 4, 2); \\ y = 6 &: (n, m, a) = (5, 2, 1), (9, 3, 2); \\ y = 10 &: (n, m, a) = (6, 3, 1), (16, 7, 3) \end{aligned}$$

- Kebli et al. (2020): Assume that the *abc*-conjecture holds. Then the Diophantine equation $F_n + F_m = y^a$ has only finitely many solutions $(n, m, y, a) \in \mathbb{N}^4$ with $n \geq m$, $y \geq 2$ and $a \geq 2$.

An integer n is close to a positive integer m, if it satisfies

$$|n - m| < \sqrt{m}.$$

Chern and Cui (2014): The only solutions $(n, m) \in \mathbb{N}^2$ of the Diophantine inequality

$$|F_n - 2^m| < 2^{m/2}$$

are $(n, m) = (2, 1), (3, 1), (4, 1), (4, 2), (5, 2), (6, 3), (7, 4), (9, 5)$

Theorem 1(H., 2022)

There are exactly 52 solutions $(n, m, a) \in \mathbb{N}^3$ to the Diophantine inequality

$$|F_n + F_m - 2^a| < 2^{a/2}, \; n \geq m \geq 1, \; a \geq 1 \tag{1.1}$$

All solutions satisfy $n \leq 42$ and $a \leq 28$.

Corollary 1

There are only 9 Lucas numbers which are close to a power of 2. Namely, the solutions $(n, a) \in \mathbb{N}^2$ of the inequality

$$|L_n - 2^a| < 2^{a/2}$$

are $(1, 1)$, $(2, 1)$, $(2, 2)$, $(3, 2)$, $(4, 3)$, $(6, 4)$, $(7, 5)$, $(10, 7)$ and $(13, 9)$

Some useful tools

Binet formula.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \forall n \geq 0$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha}$ are the roots of the characteristic polynomial $X^2 - X - 1$ of the Fibonacci sequence. Moreover,

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \forall n \geq 1$$

Binet formula yields for $n > 1$ the inequalities

$$0.38\alpha^n < \alpha^n \frac{1-\alpha^{-4}}{\sqrt{5}} \leq F_n = \alpha^n \frac{1-(-1)^n \alpha^{-2n}}{\sqrt{5}} \leq \alpha^n \frac{1-\alpha^{-6}}{\sqrt{5}} < 0.48\alpha^n$$

Assume that $n > m > 1$. From the above inequalities we get

$$0.38\alpha^n < F_n < F_n + F_m < 0.48\alpha^n + 0.48\alpha^{n-1} < 0.78\alpha^n. \tag{1.2}$$

Linear forms in logarithms.

Definition (Logarithmic height)

Let α be an algebraic number of degree $d \geq 1$ with the minimal polynomial

$$a_d X^d + \ldots + a_1 X + a_0 = a_d \prod_{i}^{d} (X - \alpha_i),$$

where $a_0, a_1, \ldots, a_d$ are relatively prime integers and $\alpha_1, \ldots, \alpha_d$ are the conjugates of α. The absolute logarithmic Weil height of α is defined as

$$h(\alpha) = \frac{1}{\alpha}\Big(\log |a_d| + \sum_{i}^{d} \log \left(\max\{|\alpha_i|, 1\} \right) \Big)$$

Theorem (Matveev, 2000)

Let $\gamma_1, \ldots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, let $b_1, \ldots, b_t \in \mathbb{Z}$ and

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \ldots A_t)$$

where

$$B \geq \max\{|b_1|, \cdots, |b_t|\}$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\} \text{ for all } i = 1, \ldots, t.$$

Reduction by continued fractions.

For a real number X, we denote by $||X|| = \min\{|x - n|,\ n \in \mathbb{Z}\}$ the distance from X to the nearest integer.

Lemma 1 (Dujella and Pethö, 1998)

Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction expansion of the irrational number γ such that $q > 6M$ and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there is no solution to the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \text{ and } w \geq \frac{\log\left(\frac{Aq}{\epsilon}\right)}{\log B}.$$

Lemma 2 (Legendre's criterion)

Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction expansion of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : \ i = 0, 1, 2, \cdots, N\}$, the inequality

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M) + 2)s^2}$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

Strategy

- We extract by a simple computer search all solutions (n, m, a) with $n < 250$.
- Using Binet formula, we rewrite $F_n + F_m$ in suitable ways. Combining with (1.1) we obtain two different linear forms in logarithms of algebraic numbers which are both nonzero and small.
- We use twice a lower bound on such nonzero linear forms in logarithms of algebraic numbers due to Matveev to bound n.
- As soon as we have found an upper bound for n, we apply the Baker-Davenport reduction method and obtain bounds of a size that can be more easily handled.
- In case Baker-Davenport method fails we use a criterion of Legendre.

Proof

We assume that $n > 250$ and we will show that there exist no solutions with $n > 250$.

Relation between n and a. Combining (1.1) and (1.2) we get

$$n\frac{\log\alpha}{\log 2} + \frac{\log 0.38}{\log 2} - 1 < a < n\frac{\log\alpha}{\log 2} + 1, \tag{1.3}$$

where $\frac{\log\alpha}{\log 2} = 0.6942\ldots$. In particular, we have $a < n$.

First linear form in logarithms. By the Binet formula, we have

$$\frac{\alpha^n + \alpha^m}{\sqrt{5}} - (F_n + F_m) = \frac{\beta^n + \beta^m}{\sqrt{5}} \tag{1.4}$$

Taking the absolute values in the above equality and combining it with (1.1), we get

$$\left| \frac{\alpha^n(1 + \alpha^{m-n})}{2^a\sqrt{5}} - 1 \right| < 2^{-\frac{a}{2}+1}. \tag{1.5}$$

Put $\Lambda_1 := \frac{\alpha^n(1+\alpha^{m-n})}{2^a\sqrt{5}} - 1$. In the first application of Matveev's theorem, we take the parameters $t = 3$ and

$$(\gamma_1, b_1) := (2, -a), (\gamma_2, b_2) := (\alpha, n), (\gamma_3, b_3) := \left(\frac{1 + \alpha^{m-n}}{\sqrt{5}}, 1 \right)$$

We get that

$$\left(\frac{a}{2} - 1 \right) \log 2 < 1.4 \times 10^{12} \times \log n \times (3 + (n - m) \log \alpha). \tag{1.6}$$

Second linear form in logarithms. Rewrite the Binet formula as follows

$$\frac{\alpha^n}{\sqrt{5}} - (F_n + F_m) = \frac{\beta^n}{\sqrt{5}} - F_m$$

Again combining the above relation with (1.1), we get

$$|1 - 2^a \cdot \alpha^{-n} \cdot \sqrt{5}| < \frac{2^{\frac{a}{2}}\sqrt{5}}{\alpha^n} + \frac{\sqrt{5}}{2\alpha^n} + \frac{\sqrt{5}}{\alpha^{n-m}} < \frac{3\sqrt{5}}{2}\max\{\alpha^{m-n}, \alpha^{a-n}\} \tag{1.7}$$

Put $\Lambda_2 := 1 - 2^a \cdot \alpha^{-n} \cdot \sqrt{5}$. In a second application of Matveev's theorem, we take the parameters $t = 3$ and

$$(\gamma_1, b_1) := (2, a), (\gamma_2, b_2) := (\alpha, -n), (\gamma_3, b_3) := (\sqrt{5}, 1)$$

We get

$$\min\{(n-a)\log\alpha, (n-m)\log\alpha\} < 2.4 \times 10^{12} \log n. \tag{1.8}$$

Case 1. $\min\{(n-a)\log\alpha, (n-m)\log\alpha\} = (n-m)\log\alpha$.
In this case by a calculation in *Sage*, we obtain

$$a < 4.5 \times 10^{28} \text{ and } n < 6.6 \times 10^{28}.$$

Case 2. $\min\{(n-a)\log\alpha, (n-m)\log\alpha\} = (n-a)\log\alpha$.
In this case we have

$$a < 2.3 \times 10^{14} \text{ and } n < 1.6 \times 10^{14}.$$

Thus, in both Case 1 and Case 2, we have

$$a < 4.5 \times 10^{28} \text{ and } n < 6.6 \times 10^{28}. \tag{1.9}$$

Reduction of the bound.

For any non-zero real number x, we have

i) $0 < x < e^x - 1$,

ii) if $x < 0$ and $|e^x - 1| < 1/2$, then $|x| < 2|e^x - 1|$.

We may assume that $n - m > 250$ and $n - a > 250$. We go back to the inequality (1.7). Since we assume that $\min\{n - m, n - a\} > 250$ by ii) we have

$$0 < \left| a\frac{\log 2}{\log \alpha} - n + \frac{\sqrt{5}}{\log \alpha} \right| < \frac{3\sqrt{5}}{\log \alpha} \cdot \alpha^{-\kappa},$$

where $\kappa = \min\{n - m, n - a\}$. We take $M = 4.5 \times 10^{28}$ (an upper bound for a) and apply Lemma 1 with

$$\gamma := \frac{\log 2}{\log \alpha}, \ \mu := \frac{\sqrt{5}}{\log \alpha}, \ A := \frac{3\sqrt{5}}{\log \alpha}, \ B := \alpha.$$

We get that $n - m < 158$ and $a < 1.3 \times 10^{16}$.

Let us now work on the inequality (1.5). We may assume that $a > 28$. By i) and ii) it becomes

$$0 < \left| a\frac{\log 2}{\log \alpha} - n + \frac{\log \phi(n-m)}{\log \alpha} \right| < \frac{4}{\log \alpha} \cdot 2^{-a/2}$$

where ϕ is defined by $\phi(t) := \sqrt{5}(1+\alpha^{-t})^{-1}$.
We take $M = 1.3 \times 10^{16}$ (an upper bound for a) and apply Lemma 1 with

$$\gamma := \frac{\log 2}{\log \alpha},\ \mu := \frac{\phi(n-m)}{\log \alpha},\ A := \frac{4}{\log \alpha},\ B := \sqrt{2}$$

for all choices $n - m \in \{1, \ldots, 158\}$ except when $n - m = 2, 6$. With the help of *Sage* we find that if (n, m, a) is a possible solution of (1.1) with $n - m \neq 2, 6$ then $a \leq 67$ and thus, $n \leq 100$. But this is a contradiction to our assumption that $n > 250$.

Special cases $n-m=2$ and 6. Note that

$$\mu = \frac{\log \phi(t)}{\log \alpha} = \begin{cases} 1 & \text{if } t=2 \\ 3-\frac{\log 2}{\log \alpha} & \text{if } t=6 \end{cases}$$

and the corresponding value of ϵ is always negative. When $n-m=2$ we get that

$$0 < |a\gamma - (n-1)| < \frac{4}{\log \alpha} \cdot 2^{-a/2} < \frac{4}{\log \alpha} \cdot 2^{-\frac{1}{2}\left(n\frac{\log \alpha}{\log 2} + \frac{\log 0.38}{\log 2} - 1\right)}$$

Recall that $a < 1.3 \times 10^{16}$. Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [1, 2, 3, 1, 2, \ldots]$ be the continued fraction of γ. A quick search using *Sage* reveals that

$$q_{35} < 1.3 \times 10^{16} < q_{36}.$$

Furthermore, $a_M := \max\{a_i : \ i = 0, 1, \ldots, 36\} = a_{17} = 134$. So by Legendre's criterion we have

$$|a\gamma - (n-1)| > \frac{1}{(a_M+2)a}.$$

Comparing the above estimates we get that $n < 187$. In a similar manner one can get $n < 187$ in the case when $n-m=6$. This again contradicts our assumption that $n > 250$.

Thank you!