30.08.22

NO IET IS MIXING

from
"IET and some special flows are not mixing" A. Katok

IET : interval exchange transf.

$I = [a,b]$ and $f : [a,b] \circlearrowleft$

we want f to be 1-1 and continuous except at finitely many points
f preserves the Lebesgue measure λ

Formally: $n > 0$ $\sigma \in \mathfrak{S}_n$

$\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ st $\sum \lambda_i = b - a$

$T_{\underline{\lambda}, \sigma} = T : [a, b] \circlearrowleft$

for $1 \leq i \leq n$ $\quad a_i = \sum_{1 \leq j \leq i} \lambda_j$

$$b_i = \sum_{1 \leq j \leq \sigma(i)} \lambda_{\sigma^{-1}(j)}$$

for $x \in I$ we define

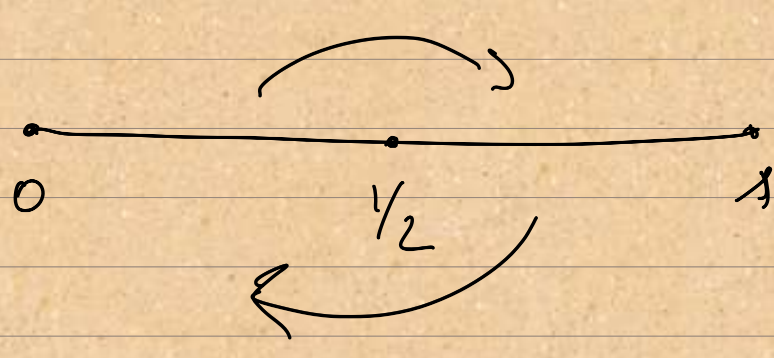
$$T(x) = x + b_i - a_i \text{ for } x \in]a_i, a_i + \lambda_i[$$

def: If m is the minimal positive integer such IET f has a representation as above we shall say that f is a IET of m intervals.

Ex

$f : [0,1] \circlearrowleft$ $\qquad \lambda = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$

$\sigma = (12)$



Rmk

In principle on IET there may be other invariant measures different from the Lebesgue one

(I, μ, f) dynamical system

def (Mixing) Let $(X, \mathcal{B}, \mu, T)$ be a dyn. syst. Then T is mixing if $\forall A, B \in \mathcal{B}$ we have

$$\lim_{n \to +\infty} \mu(T^{-n} A \cap B) = \mu(A)\mu(B)$$

<u>Theorem</u> (A. Katok)

$f : I = [a,b] \circlearrowleft$ is a IET

μ is any not atomic Borel measure on I
which is f-invariant.

f is <u>not mixing</u>.

<u>Basic idea</u>: if f were mixing

then for $A = B$ one would have

$$\lim_{n\to+\infty} \mu(f^{-n}A\cap A) = \mu(A)^2$$

the idea becomes to find a sequence $\{t_n\} \subset \mathbb{N}$ s.t.

$$\mu(A\cap f^{-t_n}A) \xrightarrow{k} \mu(A)^2$$

In order to prove this thm we need 2 lemmata.

<u>Lemma</u>: $f: I \circlearrowleft$ is an IET of m intervals and μ is a non-atomic Borel measure inv. under f

Then there exist an IET of $r \leq m$ interval $g: [0,1] \circlearrowleft$ R.

st $([0,1],\lambda) \overset{\sim}{=} (\mathcal{I},\mu)$

there exist a bijection between them up to subsets of measure 0

R is the "isomorphism"

$R: \mathcal{I} \longrightarrow [0,1]$ and this can be taken to be $\nearrow$

<u>Sketch</u>

$R: [a,b] = \mathcal{I} \longrightarrow [0,1]$

$y \longmapsto \mu([a,y])$

since μ is <u>not-atomic</u> R is continuous and surjective $\nearrow$

Generally this is not 1-1.

$$R_* \mu = \lambda$$

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "R"'] \arrow[dr, phantom, "G"] & I \arrow[d, "R"] \\ {[0,1]} \arrow[r, dashed, "g"] & {[0,1]} \end{tikzcd}$$

x

$$g(x) = R(f(y)) \qquad x = Ry$$

Some checks imply that g is an IET

#

<u>Lemma 3</u>

$f : I \circlearrowleft$ is an IET of m int.

$\Delta \subset I$ and let f_Δ be the induced IET.

$y \in \Delta \quad \tau(y) = \min\{k \mid \text{st } f^k(y) \in \Delta\}$

time of first return to Δ

f_Δ is an IET of at most $m+2$ intervals (say s). Moreover

$\Delta = \Delta_1 \cup \dots \cup \Delta_s$

$2 \leq s \leq m+2$

Proof of Thm 1

- we consider ergodic measures
- Lemma 1 $\Longrightarrow$ it is sufficient to prove the Thm

for λ on $[0,1]$

Fix $\Delta \subset I$

Lemma 2 $\Longrightarrow$ $I = \bigcup_{i=1}^{s} \bigcup_{k=0}^{t_i-1} f^k \Delta_i$

where t_i is the time of first return to Δ_i

for each Δ_i we have f_{Δ_i}

the induced IET on Δ_i

and we apply Lemma 2 once more to Δ_i

$$\Delta_i = \bigcup_{j=1}^{s_i} \Delta_{ij} = \bigcup_{l=1}^{s_i} f^{t_{ij}} \Delta_{ij}$$

first time of return to Δ_{ij}

$$I = \bigcup_{i=1}^{s} \bigcup_{j=1}^{s} \bigcup_{n=0}^{t_{ij}-1} f^n \Delta_{ij}$$

these are all disjoint intervals

Not. $f^n \Delta_i = \Delta_i^n$

$\Delta_{ij}^n \subset f^{-t_{ij}}(\Delta_i^n)$

Rmk: $f^{t_{ij}}(\Delta_{ij}^n) \subset \Delta_i^n$ $(*)$

$$\underbrace{f^{t_{ij}}(f^n \Delta_{ij})}_{\subset \Delta_i^n} = f^n \big(\underbrace{f^{t_{ij}} \Delta_{ij}}_{\in \Delta_i}\big)$$

$$\Rightarrow \Delta_i^n = \bigcup_{j=1}^{s_i} \Delta_{ij}^n$$

$$(*) \Rightarrow \Delta_i^n \subset \bigcup_{j=1}^{n} f^{-t_{ij}} \Delta_i^n$$

Recall

$$I = \bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} \bigcup_{n=0}^{t_{ij}-1} f^n \Delta_{ij}$$

this is called "portion of I"

ξ_Δ. We say the $A \subset I$ is measurable wrt ξ_Δ if A is union of element in ξ_Δ

Let A measurable wrt ξ_Δ

$$A \subset \bigcup_{i=1}^{s} \bigcup_{j=1}^{s_i} f^{-t_{ij}} A$$

f is measure function

$s \leq m+2$, $s_i \leq m+2$

$\exists\, t_{ij}$ st

- $\mu(A \cap f^{t_{ij}} A) =$

$\circledast$ $\mu(A \cap f^{-t_{ij}} A) > \frac{1}{(m+2)^2} \mu(A)$

- Fix A st $\mu(A) < \frac{1}{10(m+2)^2}$

- Fix $N > N$

Choose $\Delta \subset I$ so that
$\exists\, A_\Delta$ measurable wrt ξ_Δ

- $\mu(A \Delta A_\Delta) < \frac{1}{10} \mu(A)^2$
- $t_i > N\ \forall i$ ← this holds for any sub interval of Δ

Pick A_Δ for some $t_{ij} > t_i > N$

$$\mu(A \cap f^{h_j} A) \geq$$

$$\geq \mu(A_\Delta \cap f^{h_j} A_\Delta) - 2\mu(A \Delta A_\Delta)$$

$$\geq \frac{1}{(m+2)^2} \mu(A_\Delta) - \frac{1}{5} \mu(A)^2$$

Remark

$$\mu(A_\Delta) > \frac{9}{10} \mu(A)$$

$$\mu(A) < \frac{1}{10(m+2)^2}$$

$$\geq \left(\frac{9}{10}\right)^2 \frac{1}{(m+2)^2} \mu(A) - \frac{1}{5} \mu(A)^2$$

$$\geq \mu(A)^2 \left(\frac{10 \cdot 9^2}{10^2} - \frac{1}{5} \right)$$

$$\underbrace{\qquad\qquad\qquad}_{>2}$$

$$> 2\mu(A)^2$$

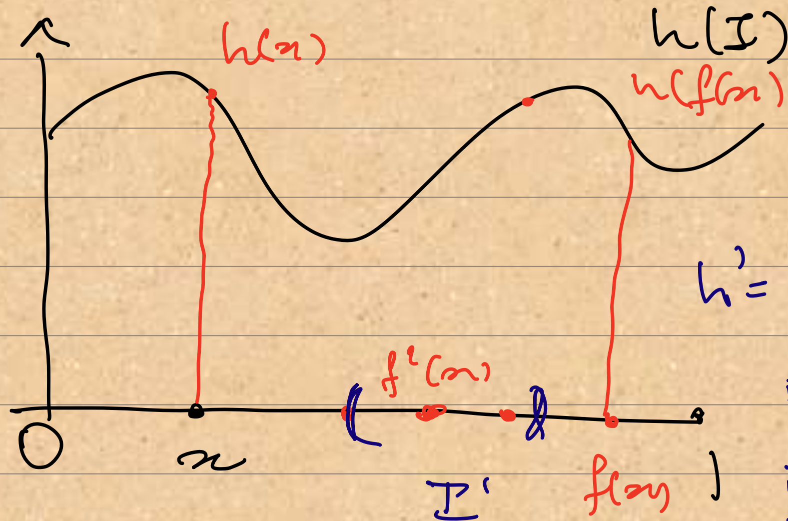
$\Longrightarrow$ f is not mixing. #

$f : I \circlearrowleft$ an IET

$h : I \longrightarrow \mathbb{R}^+$ "roof function"

$\leadsto$ determines a "vertical" flow

$\{f_t^h\}$ on $I_h = \{(x,t) \in I \times \mathbb{R} \mid 0 \le t \le h(x)\}$



$h' = h(x) + h(f(x))$

if the time of return to I' is t

$$h'(x) = \sum_{k=0}^{t-1} h(f^k(x))$$

h is bounded then any finite Borel measure μ on I determine a measure $\nu = \mu \times \lambda$ on I_h

<u>def</u>: $h \in C([a,b])$

$$V(h) = \sup_{P \in \mathcal{P}} \sum_{0 \le i \le n_p} |h(x_{i+1}) - h(x_i)|$$

where $\mathcal{P} = \{ P = (x_0 \dots x_{n_p}) \}$

$a = x_0 < x_1 < x_2 \cdots < x_{n_p} = b \}$

$h \in BV([a,b])$ if $V(h) < +\infty$

bounded variation

Thm (A. Katok)

$f : I \circlearrowleft$ be an IET

$h \in BV(h)$ ν a Borel measure non-atomic

inv. wrt $\{f_t^h\}$ "vertical flow"

Then $\{f_t^h\}$ is not mixing.