Almost-Prime Times in Horospherical Flows
West Coast Dynamics Seminar

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Homogeneous Dynamics

- $G$, a Lie group
- $\Gamma \leq G$, a lattice (discrete, finite covolume subgroup)
- $X = \Gamma \backslash G$, space of interest
- $H \leq G$, a closed subgroup
- Dynamics: $H \curvearrowright X$ by right translations

Possible questions:
- Given $x \in X$, what does the orbit $xH$ look like?
- What does a typical orbit look like?
- What $H$-invariant/ergodic measures are supported on this space?
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Example: Linear Flows on the Torus

\[ G = \mathbb{R}^2, \quad \Gamma = \mathbb{Z}^2, \quad X = \mathbb{T}^2, \quad H = \{tv \mid t \in \mathbb{R}\} \text{ for some } v \in \mathbb{R}^2 \]

- If \( v \) has rational slope, then every orbit is periodic.
- If \( v \) has irrational slope, then every orbit is dense.
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\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{torus_diagram}
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The Space of Lattices

- $G = \text{SL}_2(\mathbb{R})$
- $\Gamma = \text{SL}_2(\mathbb{Z})$
- $G \curvearrowright \mathbb{H}^2 := \{z = x + iy \in \mathbb{C} \mid y > 0\}$ by Möbius transformations:
  $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}$$

- $G \curvearrowright T^1\mathbb{H}^2$ by $g : (z, v) \mapsto (g(z), Dg v)$ with $\text{Stab}_G(z) = \{ \pm I \}$
- $\text{PSL}_2(\mathbb{R}) \cong T^1\mathbb{H}^2$
The Space of Lattices

PSL\(_2(\mathbb{Z}) \backslash PSL\(_2(\mathbb{R})\)

\[
\frac{dx dy d\theta}{y^2}
\]

\[G = SL_n(\mathbb{R}), \quad \Gamma = SL_n(\mathbb{Z}), \quad \Gamma \backslash G \cong \{\text{lattices in } \mathbb{R}^n \text{ of covolume 1}\}\]
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Subgroup Actions

\[ A = \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}_{t \in \mathbb{R}} \]

geodesic flow

\[ U = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t \in \mathbb{R}} \]

horocycle flow

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Note: \( a_t^{-1} u_s a_t = \begin{pmatrix} 1 & se^{-t} \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) as \( t \rightarrow \infty \)

**Definition**

A subgroup \( H \leq G \) is called *horospherical* if there exists \( g \in G \) such that

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H = \{ h \in G \mid g^{-n} h g^n \rightarrow e \text{ as } n \rightarrow \infty \}.
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Horospherical Subgroups

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Fact: horospherical $\not\Rightarrow$ unipotent

Example (Heisenberg group)

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\left\{ \begin{pmatrix} 1 & x & y \\ 1 & z \\ 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}
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with respect to, e.g.,

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\begin{pmatrix} 2 & 1 \\ & & 1/2 \end{pmatrix}
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Example

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\left\{ \begin{pmatrix} 1 & t & t^2/2 \\ 1 & t \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}
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Roughly speaking, a subset of $X$ \textit{equidistributes} respect to a measure $\mu$ if it spends the expected amount of time in measurable subsets.

**Example**

A sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ equidistributes with respect to $\mu$ if

$$\frac{1}{N} \sum_{n=1}^{N} f(x_n) \to \int_{X} f \, d\mu$$

for all $f \in C_c^\infty(X)$.

Say equidistribution is \textit{effective} if the rate of convergence is known.
Roughly speaking, a subset of $X$ *equidistributes* respect to a measure $\mu$ if it spends the expected amount of time in measurable subsets.

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Roughly speaking, a subset of $X$ \textit{equidistributes} respect to a measure $\mu$ if it spends the expected amount of time in measurable subsets.

\textbf{Example}

A path $\{x(t)\}_{t \in \mathbb{R}^+} \subset X$ equidistributes with respect to $\mu$ if

$$\frac{1}{T} \int_0^T f(x(t)) dt \to \int_X f \, d\mu$$

for all $f \in C_c^\infty(X)$.

Say equidistribution is \textit{effective} if the rate of convergence is known.
**Theorem**

Let $H \leq G$ be horospherical. For any $x \in X$, there exists a closed, connected subgroup $H \leq L \leq G$ such that $xH = xL$ and such that $xL$ supports an $L$-invariant probability measure $\mu_x$ with respect to which the $H$-orbit of $x$ equidistributes.

- Hedlund, Furstenberg ($SL_2$)
- Burger ($SL_2$, $\Gamma$ cocompact, effective w/ polynomial rate)
- Veech, Ellis-Perrizo (general horospherical, $\Gamma$ cocompact)
- Margulis, Dani, Dani-Margulis (quantitative nondivergence)
- Dani (above theorem)
- Strömbergsson, Flaminio-Forni ($SL_2$, $\Gamma$ non-uniform, effective w/ polynomial rate depending on basepoint)
Rigidity of Horospherical Actions

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Theorem (Dani)

For every $x = \Gamma g \in X$, either

$$\frac{1}{|B_T|} \int_{B_T} f(xu)\,du \xrightarrow{T\to\infty} \int_X f\,dm \quad \forall f \in C^\infty_c(X)$$

(1)

or there is a proper, nontrivial rational subspace $W \subset \mathbb{R}^n$ such that $Wg$ is $U$-invariant.

- $du$ Haar measure on $U$
- $dm$ pushforward of Haar measure on $G$ to $X$
- $B_T = a_{\log T}B^U_1 a_{\log T}^{-1}$ expanding Følner sets
- If $x$ satisfies (1), call it *generic*.
  (Birkhoff’s Theorem $\iff$ almost every $x$ is generic.)
Effective Equidistribution

**Theorem (M.)**

*There exists \( \gamma > 0 \) such that for every \( x = \Gamma g \in X \) and \( T > R \) large enough, either:*

\[
\left| \frac{1}{|B_T|} \int_{B_T} f(xu) du - \int_X f dm \right| \ll_f R^{-\gamma} \quad \forall f \in C^\infty_c(X) \quad (2a)
\]

*or*

\[
\exists j \in \{1, \cdots, n-1\} \text{ and } w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\} \text{ such that } \|wg_0u\| < R \quad \forall u \in B_T. \quad (2b)
\]

- If \( x \) satisfies (2a) for fixed \( R \) and all large \( T \), call it \( R \)-generic.
  Note: \( x \) is generic \( \iff \) \( x \) is \( R \)-generic for all \( R > 0 \).

- Condition (2b) says that there is a rational subspace \( W \in \mathbb{R}^n \) such that \( Wg \) is \( R \)-almost invariant when flowed up to time \( T \).
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Why do we want effective results?
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- Applications in number theory often require effective rates.
Möbius Disjointness

Recall: the Möbius function

$$\mu(n) = \begin{cases} 
0 & \text{if } n \text{ is not squarefree} \\
(-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes}
\end{cases}$$

Conjecture (Sarnak)

$$\frac{1}{N} \sum_{n \leq N} \mu(n) f(T^nx) \to 0$$

for any:

- $X$ compact metric space
- $x \in X$
- $T : X \to X$ continuous, zero topological entropy
- $f \in C(X)$
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Partial results:

- Vinogradov/Davenport (circle rotations/translations on a compact group—effective)
- Green-Tao (nilflows—effective)
- Bourgain-Sarnak-Ziegler/Peckner (unipotent flows on homogeneous spaces—not effective)
Conjecture (Margulis)

Let \( \{u_t\}_{t \in \mathbb{R}} \) be a unipotent flow on a homogeneous space \( X \). If \( \{xu_t \mid t \in \mathbb{R}\} \) equidistributes in \( X \), then so does \( \{xu_p \mid p \text{ is prime}\} \).

Theorem (Bourgain)

For any measurable dynamical system \((X, \mathcal{B}, \mu, T)\) and \( f \in L^2(X, \mu) \), the ergodic averages over primes

\[
\frac{1}{\pi(N)} \sum_{\substack{p \leq N \atop p \text{ prime}}} f(T^p x)
\]

converge for \( \mu \)-a.e \( x \in X \).
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The Horocycle Flow at Almost-Prime Times

**Definition**
An integer is called *almost-prime* if it has fewer than a fixed number of prime factors.

**Theorem (Sarnak-Ubis)**
There exists $\ell \in \mathbb{N}$ such that for any generic $x \in \text{SL}_2(\mathbb{Z})\backslash\text{SL}_2(\mathbb{R})$, the set
$$\{ xu(k) \mid k \in \mathbb{Z} \text{ has fewer than } \ell \text{ prime factors} \}$$
is dense in $\text{SL}_2(\mathbb{Z})\backslash\text{SL}_2(\mathbb{R})$. 
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Let $G = \text{SL}_n(\mathbb{R})$, $\Gamma \leq G$ a lattice, and $u(t)$ a $d$-dimensional horospherical flow on $X = \Gamma \backslash G$. Define

$$\mathcal{A}_\ell(x) = \{xu(k_1, k_2, \cdots, k_d) \mid k_i \in \mathbb{Z} \text{ has fewer than } \ell \text{ prime factors}\}.$$

**Theorem (M.)**

1. If $\Gamma$ is cocompact, then there exists $\ell = \ell(n, d, \Gamma)$ such that for any $x \in X$, the set $\mathcal{A}_\ell(x)$ is dense in $X$.

2. If $\Gamma = \text{SL}_n(\mathbb{Z})$ and $x = \Gamma g \in X$ satisfies a Diophantine property with parameter $\delta$, then there exists $\ell = \ell(n, d, \delta)$ such that $\mathcal{A}_\ell(x)$ is dense in $X$. 
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Questions?
Proof Idea:

1. Prove effective equidistribution of the continuous horospherical flow
2. Use this to prove effective equidistribution of arithmetic progressions of times
3. Apply sieve methods to deduce a statement about almost-primes
Effective Equidistribution of the Continuous Flow

Proof Idea: Margulis’s thickening method

\[ xu(s) \rightarrow xu(1) \rightarrow xu(T) \]

\[ xa_{\log T} u(s) a_{\log T}^{-1} \]
Effective Equidistribution of the Continuous Flow

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Effective Equidistribution of the Continuous Flow

Effective mixing of the $A$-action:

**Theorem (Howe-Moore, Kleinbock-Margulis)**

Let $\Gamma$ be cocompact. There exists $\tilde{\gamma} > 0$ such that for any $x \in X$ and $f, g \in C_c^\infty(X)$,

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\left| \int_X f(xa_t)g(x)dm - \int_X f \, dm \int_X g \, dm \right| \ll_{f,g} e^{-\tilde{\gamma}t}.
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Note:

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\frac{1}{|B_T|} \int_{B_T} f(xu)du = \int_{B_1} f(xa_{\log T}ua_{\log T}^{-1})du = \int_U \chi_{B_1}(u)f(xa_{\log T}ua_{\log T}^{-1})du
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Effective Equidistribution of the Continuous Flow

\[ \int_U \chi_{B_1}(u)f(xa_{\log T}u a_{\log T}^{-1})du \]

Problems:

1. \( \chi_{B_1} \) not smooth
   - Convolve with a smooth approximation to the identity

2. Integral over \( U \), not \( X \)
   - Thicken to get integral in \( G \), project to \( X \) (need to make sure it injects)

3. Moving basepoint
   - Quantitative nondivergence (Dani-Margulis) \( \implies \) can get a good radius of convergence for all but a small proportion of \( u \in B_1 \)
Effective Equidistribution of the Continuous Flow

\[ \int_U \chi_{B_1}(u) f(xa_{\log T}ua_{\log T}^{-1}) \, du \]

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Theorem (M.)

Let \( u(t_1, \cdots, t_d) \) be an abelian horospherical flow. There exists \( \beta > 0 \) such that if \( x \in X \) satisfies (2a) for \( T > R \) large enough, then for any \( 1 \leq K \leq T \) we have

\[
\left| \frac{K^d}{T^d} \sum_{\substack{k \in \mathbb{Z}^d \\ni Kk \in B_T}} f(xu(Kk)) - \int_X f \, dm \right| \ll_f R^{-\beta} K^{d/(d+1)} S(f).
\]
Proof Idea: Venkatesh’s van der Corput method

For simplicity, assume $G = \text{SL}_2(\mathbb{R})$, $\int f \, dm = 0$.

Let

$$E_{K,T}(f) = \frac{K}{T} \sum_{k \in \mathbb{Z}} \text{mod}_{0 \leq Kk < T} f(xu(Kk))$$

be the average over the set:

$$x \quad xu(K) \quad xu(2K) \quad \ldots \quad xu(T)$$
Define new function for $1 < H < T$:

$$f_H(x) = \frac{1}{H} \sum_{\ell=0}^{H} f(xu(K\ell))$$

Note: $E_{K,T}(f_H)$ is close to $E_{K,T}(f)$:
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Thicken the discrete set in $U$ by $\delta > 0$:

$$x \quad xu(K) \quad xu(2K) \quad \cdots \quad xu(T)$$

Let $E_{K,T,\delta}$ be the ergodic average over this set.

Note: By uniform continuity, $E_{K,T,\delta}(f_H)$ is close to $E_{K,T}(f_H)$. 
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Note:

\[ E_{K,T,\delta}(f_H)^2 \ll \frac{K}{\delta H^2} \sum_{\ell_1=0}^{H} \sum_{\ell_2=0}^{H} \frac{1}{T} \int_0^T f(xu(s)u(K\ell_1))f(xu(s)u(K\ell_2))ds \]
Effective Equidistribution of Arithmetic Progressions

Note:

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↓ effective equidistribution

\[ \ll \frac{K}{\delta H^2} \sum_{\ell_1=0}^{H} \sum_{\ell_2=0}^{H} \langle u(K(\ell_1 - \ell_2)) \cdot f, f \rangle_{L^2(X)} + error \]
Effective Equidistribution of Arithmetic Progressions

Note:

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\[ \downarrow \text{effective equidistribution} \]

\[ \ll \frac{K}{\delta H^2} \sum_{\ell_1=0}^{H} \sum_{\ell_2=0}^{H} \langle u(K(\ell_1 - \ell_2)) \cdot f, f \rangle_{L^2(X)} + \text{error} \]

\[ \downarrow \text{bounds on matrix coefficients} \]

\[ \ll \frac{K}{\delta H^2} \sum_{\ell_1=0}^{H} \sum_{\ell_2=0}^{H} (1 + K|\ell_1 - \ell_2|)^{-a} S(f)^2 + \text{error} \]
Choose $H$, $\delta$ to optimize the various error terms.
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Choose \(H, \delta\) to optimize the various error terms
To sieve the orbits for almost-primes, need control over averages along arithmetic progressions—this is exactly what the last theorem tells us.

For \( f \in C_c^\infty(X) \) and \( T \) large enough,

\[
\frac{\left(\log T\right)^d}{T^d} \sum_{k \in B_T, (k_1 \cdots k_d, P) = 1} f(xu(k)) \asymp \alpha \int f \, dm
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where \( P \) is the product of primes less than \( T^\alpha \).

Note: The lower bound implies the result for integer points with fewer than \( 1/\alpha \) prime factors (consider \( f \) a bump function on any small set).
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Sieving

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Thank you!