Number Theory versus Random Matrix Theory

The joint moments story

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Section

$\zeta(s)$ at the relative extrema on the critical line

Conrey and Ghosh (1985) proved under the Riemann Hypothesis, that for γ, γ^+ consecutive non-trivial zeros of $\zeta(s)$,

$$\sum_{0<\gamma\leq T} \max_{\gamma\leq t\leq\gamma^+} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \sim \frac{e^2-5}{4\pi} T(\log T)^2$$

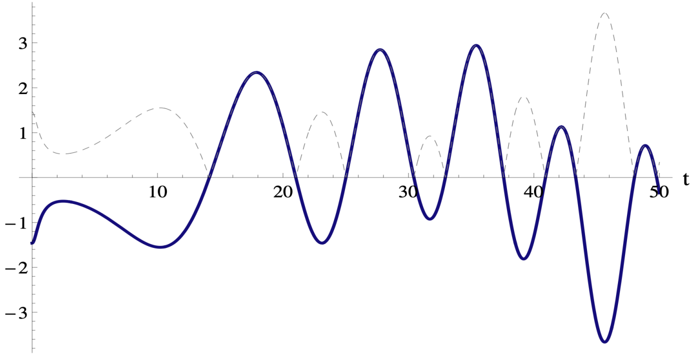
at $T \to \infty$.

Hardy Z-Function

The Hardy Z-function is defined by

$$Z(t) = \sqrt{\chi\left(\tfrac{1}{2} - it\right)}\zeta(\tfrac{1}{2} + it)$$

Then $Z(t)$ is real for real t, $Z(t)$ changes sign whenever there is a zero of odd order of $\zeta(s)$, and $|Z(t)| = |\zeta(1/2 + it)|$.



A plot of $Z(t)$ (dark blue) and $|\zeta(1/2 + it)|$ (dashed) for $0 \leq t \leq 50$

Moments of the Z-function

We can rewrite Conrey and Ghosh result as

$$\sum_{\substack{0<\lambda\leq T \\ Z'(\lambda)=0}} Z(\lambda)^2 \sim \frac{e^2-5}{4\pi} T(\log T)^2$$

at $T \to \infty$.
In fact, this can be rewritten in terms of integral moments of $|Z(t)Z'(t)|$, given by

$$\int_1^T |Z(t)Z'(t)| \, dt = \sum_{\substack{0<\lambda\leq T \\ Z'(\lambda)=0}} Z(\lambda)^2 \sim \frac{e^2-5}{4\pi} T(\log T)^2$$

at $T \to \infty$.

Moments of the Z-function

More generally, Milinovich showed under the Riemann Hypothesis, that for γ, γ^+ consecutive non-trivial zeros of $\zeta(s)$ and for k a positive integer,

$$\sum_{0<\gamma\leq T} \max_{\gamma\leq t\leq\gamma^+} \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} = \sum_{\substack{0<\lambda\leq T \\ Z'(\lambda)=0}} Z(\lambda)^{2k}$$

$$= \int_1^T |Z(t)|^{2k-1}|Z'(t)|\ dt$$

By showing upper and lower bounds of the correct order, he conjectures

$$\int_1^T |Z(t)|^{2k-1}|Z'(t)|\ dt \sim C_k T(\log T)^{k^2+1}$$

as $T \to \infty$, but doesn't conjecture a value of C_k.

Joint Moments

Consider a more general joint moment, written as

$$\int_1^T |Z(t)|^{2s-2h}|Z'(t)|^{2h}\, dt \sim F(s,h)A(s)T(\log T)^{s^2+2h}$$

as $T \to \infty$, where $A(s)$ is an arithmetic factor.
Using Random Matrix Theory, Hughes conjectured a value of $F(s,h)$ for integer s, h which agreed with all known Number Theory results.

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If we could set $h = 1/2$, then we could infer Milinovich's conjecture with the constant given explicitly.

Alternative approaches to Joint Moments

Other approaches to finding $F(s, h)$ include, for s, h integer,

- Conrey, Rubinstein and Snaith (in the special case where $s = h$)
- Dehaye
- E. Basor, P. Bleher, R. Buckingham, T. Grava, A. Its, E. Its, and J. P. Keating
- E. C. Bailey, S. Bettin, G. Blower, J. B. Conrey, A. Prokhorov, M. O. Rubinstein, and N. C. Snaith
- Forrester and Witte

Joint Moments for non-integer s, h

Winn established results for $s \in \mathbb{N}$ and $h \in \frac{1}{2}\mathbb{N}$ by finding connections with hypergeometric functions, giving an expression for $F(s, h)$ in terms of a combinatorial sum.
In particular, he showed for the RMT version of the joint moments,

$$F(1, 1/2) = \frac{e^2 - 5}{4\pi}$$

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Assiotis, Keating and Warren proved results on the random matrix side for arbitrary real values of $s > -1/2$ and positive real values of h in the full range $0 < h < s + 1/2$.

Section

A full asymptotic for Conrey and Ghosh

Under the Riemann Hypothesis, for γ, γ^+ consecutive non-trivial zeros of $\zeta(s)$ with $L = \log \frac{T}{2\pi}$. Then for any fixed $N \geq 0$, as $T \to \infty$ we have

$$\sum_{0<\gamma\leq T} \max_{\gamma\leq t\leq\gamma^+} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 = \frac{e^2-5}{2}\frac{T}{2\pi}L^2 + \alpha_{-1}\frac{T}{2\pi}L + \frac{T}{2\pi}\sum_{n=0}^{N}\frac{\alpha_n}{L^n} + O_N\left(\frac{T}{L^{N+1}}\right)$$

where

$$\alpha_{-1} = 5 - e^2 - 10\gamma_0 + 2e^2\gamma_0$$
$$\alpha_0 = 12\gamma_1 - 4e^2\gamma_1 - 5 + e^2 + 10\gamma_0 - 2e^2\gamma_0 - 4\gamma_0^2$$

and for $n \geq 1$

$$\alpha_n = \sum_{k=n}^{\infty}\frac{2^{k+1}c_{k,-k-1+n}}{(k-n)!} + (n-1)!\sum_{k=1}^{\infty}\frac{2^{k+1}}{(k-1)!}\sum_{j=1}^{\min\{k+1,n\}}\binom{k}{j-1}c_{k,-k-2+j}$$

where the $c_{k,\ell}$ are the Laurent series coefficients around $s=1$ of

$$\left(\frac{\zeta'}{\zeta}(s)\right)'\left(-\frac{\zeta'}{\zeta}(s)\right)^{k-1}\zeta^2(s)\frac{1}{s} = \sum_{\ell=-(k+3)}^{\infty}c_{k,\ell}(s-1)^{\ell}.$$

Outline of the Proof

Step 1: Consider an auxiliary function given by

$$Z_1(s) = \zeta'(s) - \frac{1}{2}\frac{\chi'}{\chi}(s)\zeta(s).$$

This function vanishes on the critical line exactly where $|\zeta(1/2 + it)|$ is a maximum.

Outline of the Proof

Step 2: Use Cauchy's Theorem to write

$$\sum_{0<\gamma\leq T} \max_{\gamma\leq t\leq\gamma^+} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{Z_1'}{Z_1}(s)\zeta(s)\zeta(1-s)\, ds,$$

where $\mathcal{C}$ is a positively oriented contour with vertices $c+i$, $c+iT$, $1-c+iT$, and $1-c+i$, where $c = 1+1/\log T$.

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where $\mathcal{C}$ is a positively oriented contour with vertices $c+i$, $c+iT$, $1-c+iT$, and $1-c+i$, where $c = 1+1/\log T$.
We can show that the top and bottom of the contour only contribute to a power-saving error term.

Outline of the Proof

Step 3: We then manipulate the RHS of the contour to write

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{Z_1'}{Z_1}(s)\zeta(s)\zeta(1-s)\ ds =$$

$$2\Re\left(\frac{1}{2\pi i}\int_{c+i}^{c+iT} \frac{Z_1'}{Z_1}(s)\chi(1-s)\zeta^2(s)\ ds\right) - \frac{1}{2\pi i}\int_{1-c+it}^{1-c+iT} \frac{\chi'}{\chi}(s)\zeta(s)\zeta(1-s)\ ds.$$

The twisted second moment can be evaluated exactly.

Outline of the Proof

Step 4: We need a series expansion for the logarithmic derivative of $Z_1(s)$, where

$$Z_1(s) = \zeta'(s) - \frac{1}{2}\frac{\chi'}{\chi}(s)\zeta(s).$$

Take the logarithmic derivative and apply the arithmetic identities

$$\sum_{d|n} \mu(d)\log(\tfrac{n}{d}) = \Lambda(n) \text{ and } \sum_{d|n} \mu(d)\log^2(\tfrac{n}{d}) = \Lambda(n)\log(n) + (\Lambda * \Lambda)(n)$$

in the region of convergence for the Dirichlet series expansions of $\zeta(s)$ and $\zeta'(s)$. This all enables us to write

$$\frac{Z_1'}{Z_1}(s) = \sum_{n=1}^{\infty} \frac{a(n,s)}{n^s}$$

Outline of the Proof

Step 5: Use stationary phase arguments to show

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \frac{Z_1'}{Z_1}(s)\chi(1-s)\zeta^2(s)\; ds =$$

$$-\sum_{nm\leq T/2\pi} \Lambda(n)d(m) + \sum_{k=1}^{\infty} 2^k \sum_{nm\leq T/2\pi} \frac{a_k(n)}{(\log nm)^k} d(m),$$

where $\Lambda(n)$ is the von Mangoldt function, $d(m)$ is the divisor function, and $a_k(n) = ((\Lambda \log) * \Lambda_{k-1})\,(n)$, with

$$\Lambda_{k-1} = \underbrace{\Lambda * \Lambda * \Lambda * \cdots * \Lambda}_{k-1 \text{ copies of } \Lambda}$$

with the convention that $\Lambda_0(n)$ takes the value 1 if $n = 1$ and 0 otherwise, and $\Lambda_1(n) = \Lambda(n)$.

Outline of the Proof

Step 6: Use Perron to evaluate the first term from the stationary phase argument

$$-\sum_{nm\leq T/2\pi}\Lambda(n)d(m)+\sum_{k=1}^{\infty}2^k\sum_{nm\leq T/2\pi}\frac{a_k(n)}{(\log nm)^k}d(m).$$

This cancels perfectly (up to a power-saving error term) with the twisted second moment

$$-\frac{1}{2\pi i}\int_{1-c+it}^{1-c+iT}\frac{\chi'}{\chi}(s)\zeta(s)\zeta(1-s)\,ds,$$

leaving

$$\sum_{0<\gamma\leq T}\max_{\gamma\leq t\leq\gamma^+}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2=\Re\left(\sum_{k=1}^{\infty}2^{k+1}\sum_{nm\leq T/2\pi}\frac{a_k(n)}{(\log nm)^k}d(m)\right)$$

(up to a power-saving error term).

Outline of the Proof

Step 7: We can use Perron on the numerator of the inner sum

$$\sum_{0<\gamma\leq T} \max_{\gamma\leq t\leq\gamma^+} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 = \Re\left(\sum_{k=1}^{\infty} 2^{k+1} \sum_{nm\leq T/2\pi} \frac{a_k(n)}{(\log nm)^k} d(m)\right)$$

to show for large x,

$$A_k(x) = \sum_{nm\leq x} a_k(n)d(m) = x\sum_{j=0}^{k+2} b_{k,j}(\log x)^{k+2-j} + O\left(x^{1/2+\epsilon}\right)$$

where

$$b_{k,j} = \frac{c_{k,-k-3+j}}{(k+2-j)!}$$

where the $c_{k,\ell}$ are the Laurent series coefficients around $s=1$ of

$$\left(\frac{\zeta'}{\zeta}(s)\right)' \left(-\frac{\zeta'}{\zeta}(s)\right)^{k-1} \zeta^2(s)\frac{1}{s} = \sum_{\ell=-(k+3)}^{\infty} c_{k,\ell}(s-1)^{\ell}$$

Outline of the Proof

Step 8a: Apply partial summation to reinsert the logarithm in the denominator of the inner sum, with $f(x) = 1/(\log x)^k$ and $L = \log T/2\pi$,

$$\begin{aligned}
&\Re\left(\sum_{k=1}^{\infty} 2^{k+1} \sum_{nm \leq T/2\pi} \frac{a_k(n)}{(\log nm)^k} d(m)\right) \\
&= \sum_{k=1}^{\infty} \left(2^{k+1} A_k\left(\frac{T}{2\pi}\right) f\left(\frac{T}{2\pi}\right) - 2^{k+1} \int_2^{T/2\pi} A_k(x) f'(x)\ dx\right) \\
&= \sum_{k=1}^{\infty} \left(2^{k+1} \frac{T}{2\pi} \sum_{j=0}^{k+2} b_{k,j} L^{2-j} + k 2^{k+1} \int_2^{T/2\pi} \left(\sum_{j=0}^{k+2} b_{k,j} (\log x)^{1-j}\right) dx\right)
\end{aligned}$$

Outline of the Proof

Step 8b: Collect coefficients of the logarithms to write

$$\frac{T}{2\pi}L^2\sum_{k=1}^{\infty}2^{k+1}b_{k,0}$$

$$+\frac{T}{2\pi}L\sum_{k=1}^{\infty}\left[2^{k+1}b_{k,1}+2^{k+1}kb_{k,0}\right]$$

$$+\frac{T}{2\pi}\sum_{k=1}^{\infty}\left[2^{k+1}b_{k,2}-2^{k+1}kb_{k,0}+2^{k+1}kb_{k,1}\right]$$

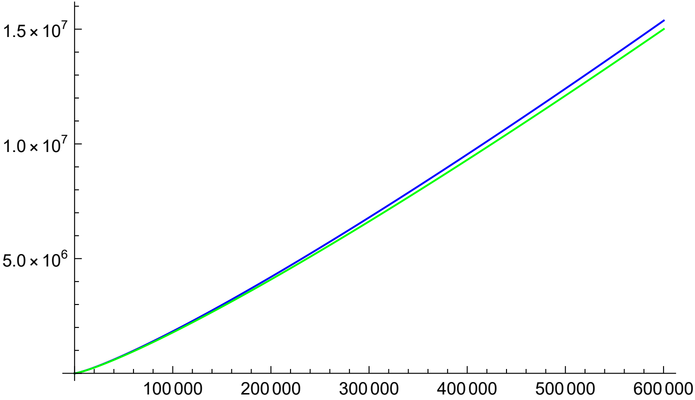
$$+\frac{T}{2\pi}\sum_{k=1}^{\infty}2^{k+1}\sum_{j=1}^{k}b_{k,j+2}\frac{1}{L^j}+\sum_{k=1}^{\infty}k2^{k+1}\sum_{j=1}^{k+1}b_{k,j+1}\int_2^{T/2\pi}\frac{1}{(\log x)^j}dx.$$

Insert the values for $b_{k,j}$ (e.g. $b_{k,0}=\frac{1}{(k+2)!}$ and $b_{k,1}=\frac{-1-(k-3)\gamma_0}{(k+1)!}$) and sum over $k\geq 1$ to obtain the asymptotic.

Section

Graphs

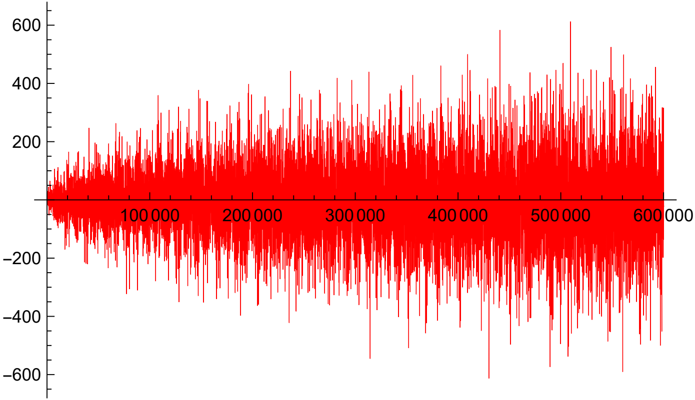
The cumulative total of $Z(\lambda)^2$ is in blue over the first million values of λ. The result of Conrey and Ghosh, $\frac{e^2-5}{2}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2$, is in green.



Graphs

The following graph shows the error between the true value of the sum and the asymptotic form given our Theorem, in the case when $N = 6$, that is

$$\sum_{0<\gamma\leq T} \max_{\gamma^-\leq t\leq\gamma^+} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 - \left(\frac{e^2-5}{2}\frac{t}{2\pi}\left(\log\frac{t}{2\pi}\right)^2 + \frac{t}{2\pi}\sum_{n=-1}^{N}\alpha_n\left(\log\frac{t}{2\pi}\right)^{-n}\right).$$



Numerics

We can see the effect of increasing N on the goodness of the asymptotic expansion. Fixing T at the height of the millionth local maximum, the true value of the cumulative sum is 1.53778×10^7, and the table shows the absolute error for varying N at that point.

N	Error
-2 (the leading order)	371166.05
-1	-33026.28
0	7412.69
1	-1072.47
2	113.86
3	-91.45
4	-54.63
5	-62.32
6	-60.88

Section

Other Results: Removing the Modulus

Under the Riemann hypothesis, for $\gamma \leq \gamma^+$ consecutive non-trivial zeros,

$$\sum_{\substack{0<\lambda\leq T \\ Z'(\lambda)=0}} \zeta\left(\tfrac{1}{2}+i\lambda\right) = \frac{e^2-3}{2}\frac{T}{2\pi}\log\frac{T}{2\pi} + \frac{3-e^2-4\gamma_0}{2}\frac{T}{2\pi} + \frac{T}{2\pi}\sum_{k=1}^{K}\frac{c_k}{(\log T/2\pi)^k} + O_K\left(\frac{T}{(\log T)^{K+1}}\right)$$

Other Results: Shanks' Style Conjecture

Shanks' conjecture states that $\zeta'(s)$, when averaged over the non-trivial zeros of $\zeta(s)$, is positive and real on average, with

$$\sum_{0<\gamma\leq T} \zeta'\left(\tfrac{1}{2}+i\gamma\right) \sim \frac{1}{2}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2$$

As an analogue, we can show under the Riemann hypothesis, for $\gamma \leq \gamma^+$ consecutive non-trivial zeros,

$$\sum_{\substack{0<\lambda\leq T \\ Z'(\lambda)=0}} \zeta'\left(\tfrac{1}{2}+i\lambda\right) \sim \frac{3-e^2}{4}\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2$$

which is negative and real on average.

Other Results: Generalised Shanks' Style Conjecture

For higher derivatives, $\zeta^{(n)}(s)$, when averaged over the non-trivial zeros of $\zeta(s)$, is positive/negative and real on average, depending on whether n is odd/even, with

$$\sum_{0<\gamma\leq T} \zeta^{(n)}(\rho) \sim \frac{(-1)^{n+1}}{n+1} \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{n+1}$$

Under the Riemann hypothesis, for $\gamma \leq \gamma^{+}$ consecutive non-trivial zeros,

$$\sum_{\substack{0<\lambda\leq T \\ Z'(\lambda)=0}} \zeta^{(n)}\left(\tfrac{1}{2} + i\lambda\right)$$

$$\sim (-1)^{n} \left(\frac{e^{2}-2}{n+1} + (-1)^{n+1} \frac{n!}{2^{n+1}} \left(1 - e^{2} \sum_{k=0}^{n+1} \frac{(-2)^{k}}{k!} \right) \right) \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{n+1}$$

Other Results: Functional Equation Factor

We can average $\chi(s)$, the factor from the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$, over the non-trivial zeros of $\zeta(s)$, which gives

$$\sum_{0<\gamma\leq T} \chi\left(\tfrac{1}{2} + i\gamma\right) = -\frac{T}{2\pi} + O\left(T^{1/2+\varepsilon}\right)$$

We can show under the Riemann hypothesis, for $\gamma \leq \gamma^+$ consecutive non-trivial zeros,

$$\sum_{\substack{0<\lambda\leq T \\ Z'(\lambda)=0}} \chi\left(\tfrac{1}{2} + i\lambda\right) = (e^2-2)\frac{T}{2\pi} + \frac{T}{2\pi}\sum_{k=1}^{K} \frac{c_k}{(\log T/2\pi)^k} + O_K\left(\frac{T}{(\log T)^{K+1}}\right)$$

A New Challenge

Can RMT retake the lead to predict full asymptotics of the type of problems discussed today?