The Riemann Hypothesis via the generalized von Mangoldt function

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(based on joint work with William Banks)

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Let $s = \sigma + it$ be a complex variable. The Riemann-zeta function is defined as:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

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Here, $\Lambda(n)$ is defined as:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^{\alpha} \text{ for } p \text{ prime and some } \alpha > 1 \\ 0 & \text{otherwise} \end{cases}$$

Previous Results

Theorem (Gonek, Graham, Lee, 2020)

A necessary and sufficient condition for the truth of the Riemann Hypothesis is that for any fixed constants $\varepsilon, B > 0$, one has the uniform estimate

$$\sum_{n \leqslant x} \Lambda(n) n^{-iy} = \frac{x^{1-iy}}{1-iy} + O(x^{1/2}|y|^{\varepsilon}) \qquad (2 \leqslant x \leqslant |y|^{B}), \tag{1.1}$$

where Λ is the von Mangoldt function.

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Theorem (von Koch, 1901)

Assume RH. Then for $x \geqslant 2$,

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n) = x + O(x^{1/2}(\log x)^2).$$

Definition

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$$\sum_{n=1}^{\infty} \Lambda_k(n) n^{-s} = (-1)^k \frac{\zeta^{(k)}}{\zeta}(s) \qquad (\sigma > 1).$$

Our Result

- We study twisted sums of the form:

$$\psi^k(x,y) := \sum_{n \leqslant x} \Lambda^k(n) n^{-iy} \tag{1.2}$$

$$\psi_k(x,y) := \sum_{n \leqslant x} \Lambda_k(n) n^{-iy} \tag{1.3}$$

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- **Goal:** To reformulate Riemann hypothesis in terms of asymptotic estimates for twisted sums with Λ^k and Λ_k.
- We prove the analogues of Gonek, Graham and Lee's result with twisted partials sums involving Λ^k and Λ_k.

Main Theorems for $\psi^k(x,y) = \sum_{n \leqslant x} \Lambda^k(n) n^{-iy}$

Theorem 1 (Banks, S., 2022)

Fix $k \in \mathbb{N}$. If the Riemann Hypothesis is true, then

$$\psi^k(x,y) = \operatorname*{Res}_{w=1-iy} \left(\left\{ -\frac{\zeta'}{\zeta}(w+iy) \right\}^k \frac{x^w}{w} \right) + O\left(x^{1/2}\{\log(x+|y|)\}^{2k+1}\right)$$

holds uniformly for all $x, y \in \mathbb{R}$, $x \geqslant 2$, where the implied constant depends only on k. The residual term can be omitted if $|y| > \sqrt{x}$, and the exponent $2k+1$ can be replaced by 2 in the case that $k = 1$.

Theorem 2 (Banks, S., 2022)

Fix $k \in \mathbb{N}$, and suppose that for any $\varepsilon > 0$ the estimate

$$\psi^k(x,y) = \operatorname*{Res}_{w=1-iy}\left(\left\{-\frac{\zeta'}{\zeta}(w+iy)\right\}^k \frac{x^w}{w}\right) + O\big(x^{1/2}(x+|y|)^{\varepsilon}\big)$$

holds uniformly for all $x, y \in \mathbb{R}$, $x \geqslant 2$, where the implied constant depends only on k and ε. Then the Riemann Hypothesis is true.

Applications

- With $k = 1$, we get:

$$\sum_{n \leqslant x} \Lambda(n) n^{-iy} = \frac{x^{1-iy}}{1-iy} + O\left(x^{1/2}\{\log(x + |y|)\}^2\right) \qquad (x, y \in \mathbb{R},\ x \geqslant 2)$$

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Applications

- With $k = 1$, we get:

$$\sum_{n\leqslant x} \Lambda(n)n^{-iy} = \frac{x^{1-iy}}{1-iy} + O\big(x^{1/2}\{\log(x+|y|)\}^2\big) \qquad (x, y \in \mathbb{R},\ x \geqslant 2)$$

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$$\psi(x) = x + O(x^{1/2}(\log x)^2).$$

- Theorem (1) (with $k := 2$) provides the conditional estimate

$$\sum_{n\leqslant x} (\Lambda\star\Lambda)(n)n^{-iy} = \frac{x^{1-iy}(\log x - 2C_0)}{1-iy} - \frac{x^{1-iy}}{(1-iy)^2} + O\big(x^{1/2}\{\log(x+|y|)\}^5\big).$$

Main Theorems for $\psi_k(x,y) = \sum_{n \leqslant x} \Lambda_k(n) n^{-iy}$

Theorem 3 (Banks, S., 2022)

Fix $k \in \mathbb{N}$. If the Riemann Hypothesis is true, then the estimate

$$\psi_k(x,y) = (-1)^k \operatorname*{Res}_{w=1-iy} \left(\frac{\zeta^{(k)}}{\zeta}(w+iy)\frac{x^w}{w} \right) + O(x^{1/2}\{\log(x+|y|)\}^{2k+1})$$

holds uniformly for all $x, y \in \mathbb{R}$, $x \geqslant 2$, where the implied constant depends only on k. The residual term can be omitted if $|y| > \sqrt{x}$, and the exponent $2k+1$ can be replaced by 2 in the case that $k = 1$.

Theorem 4 (Banks, S., 2022)

Fix $k \in \mathbb{N}$, and suppose that for any $\varepsilon > 0$ the estimate

$$\psi_k(x, y) = (-1)^k \operatorname*{Res}_{w=1-iy} \left(\frac{\zeta^{(k)}}{\zeta}(w + iy) \frac{x^w}{w} \right) + O(x^{1/2}(x + |y|)^{\varepsilon})$$

holds uniformly for all $x, y \in \mathbb{R}$, $x \geqslant 2$, where the implied constant depends only on k and ε. Then the Riemann Hypothesis is true.

Applications

- Theorem (3) (with $k := 2$) asserts that the conditional estimate

$$\sum_{n \leqslant x} \Lambda_2(n) n^{-iy} = \frac{2x^{1-iy}(\log x - C_0)}{(1-iy)} - \frac{2x^{1-iy}}{(1-iy)^2} + O\big(x^{1/2}\{\log(x+|y|)\}^5\big)$$

  holds uniformly for all $x, y \in \mathbb{R}$, $x \geqslant 2$.

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holds uniformly for all $x, y \in \mathbb{R}$, $x \geqslant 2$.

- In particular, under RH we have

$$\sum_{n \leqslant x} \Lambda_2(n) = 2x(\log x - C_0 - 1) + O\left(x^{1/2}(\log x)^5\right).$$

Implications

- The distribution of primes is influenced by the zeros of zeta function, but our results suggest that these zeros also influence the distribution of almost-primes.

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- The distribution of primes is influenced by the zeros of zeta function, but our results suggest that these zeros also influence the distribution of almost-primes.
- We expect that these results also hold for a wider class of arithmetic functions.

Proof of Theorems 1 and 3

- Let $x, y \in \mathbb{R}$ with $x \geqslant 2$. Let

$$\sigma_0 := 1 + 1/\log x \qquad \text{and} \qquad T \in \left[\sqrt{x} + 10, \sqrt{x} + 11\right].$$

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- We use Perron's formula:

$$\sum_{n \leqslant x} a_n(y) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(y, s) \frac{x^s}{s} \, ds + O(E). \tag{1.4}$$

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- For Thm 1, $a_n(y) = \Lambda^k(n) n^{-iy}$ and $\alpha(y, s) = (-1)^k \left\{ \frac{\zeta'}{\zeta}(s) \right\}^k$.

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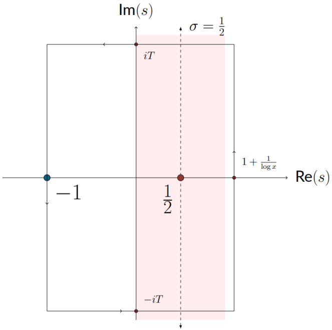
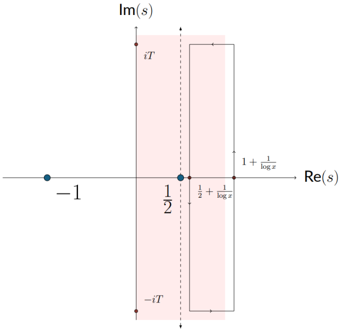
- For Thm 1, $a_n(y) = \Lambda^k(n) n^{-iy}$ and $\alpha(y, s) = (-1)^k \left\{ \frac{\zeta'}{\zeta}(s) \right\}^k$.
- For Thm 3, $a_n(y) = \Lambda_k(n) n^{-iy}$ and $\alpha(y, s) = (-1)^k \frac{\zeta^{(k)}}{\zeta}(s)$.

- We split into two cases:
  1. For Case $k = 1$, we choose the contour $\mathscr{C}$ in $\mathbb{C}$ that connects
  $$\sigma_0 - iT \longrightarrow \sigma_0 + iT \longrightarrow -1 + iT \longrightarrow -1 - iT \longrightarrow \sigma_0 - iT.$$
  2. For Cases $k \geqslant 2$, we choose the contour $\mathscr{C}$ in $\mathbb{C}$ that connects:
  $$\sigma_0 - iT \longrightarrow \sigma_0 + iT \longrightarrow \frac{1}{2} + \frac{1}{\log x} + iT \longrightarrow \frac{1}{2} + \frac{1}{\log x} - iT \longrightarrow \sigma_0 - iT.$$



Case $k = 1$



Case $k \geq 2$

Bounds on $\zeta^{(k)}/\zeta$ and $(\zeta'/\zeta)^k$

Proposition 1.1 (For $k \geqslant 2$)

Assume RH. For any $k \in \mathbb{N}$ and $x \geqslant 2$, the bounds

$$\left\{\frac{\zeta'}{\zeta}(s)\right\}^k \ll \left(\log(x\tau)\log\tau\right)^k \tag{1.5}$$

and

$$\frac{\zeta^{(k)}}{\zeta}(s) \ll \left(\log(x\tau)\log\tau\right)^k \tag{1.6}$$

hold uniformly throughout the region

$$\mathcal{R}_x := \left\{s \in \mathbb{C} : \tfrac{1}{2} + \tfrac{1}{\log x} \leqslant \sigma \leqslant 2,\ |s-1| > \tfrac{1}{100}\right\}.$$

Proof of Theorems 2 and 4

- Denote
$$\Psi(x, y) := \sum_{n \leqslant x} a_n(y)$$
(hence $\Psi = \psi^k$ or ψ_k) and
$$R(x, y) := \Psi(x, y) - \operatorname*{Res}_{w=1-iy} \left(\alpha(y, w) \frac{x^w}{w} \right).$$

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- Our hypothesis (in both theorems) is that
$$R(x,y) \ll x^{1/2}(x+|y|)^{\varepsilon}. \tag{1.7}$$

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We want to show if (1.7) holds, then RH is true.

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- Let H be the meromorphic function defined in half-plane $\sigma > 2$ by

$$H(s) := \int_1^\infty R(x, y) x^{-s} \, dx.$$

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- Let m be the multiplicity of ρ_0, and define

$$h(s) := \frac{(s-2+iy)^k \zeta(s-1+iy)^k}{(s-1+iy-\rho_0)^{mk-k_\star(\rho_0)+1}(s+iy+1)^{4k}}.$$

where

$$k_\star(\rho) := \text{the order of the pole of } H(s) \text{ at } s-1+iy = \rho_0.$$

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  2. The product $H(s)h(s)$ has no pole in the half-plane $\sigma > 1$ other than a simple pole at $s = \rho_0 + 1 - iy$.
  3. Some other things ...
- We calculate the integral below in two different ways-

$$\frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} h(s)H(s)e^{s \log x}\, ds.$$

- **First Way:** By shifting the line of integration to $\sigma = \frac{5}{4}$.

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} h(s)H(s)e^{s\log x}\,ds = c\,x^{\rho_0 - iy+1} + O(x^{5/4}), \qquad (1.8)$$

where $c\,x^{\rho_0-iy+1}$ is the residue at $s = \rho_0 + iy - 1$.

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where $c\,x^{\rho_0-iy+1}$ is the residue at $s = \rho_0 + iy - 1$.

- **Second Way:** By changing the order of integration:

$$\frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} h(s)H(s)e^{s\log x}\,ds \ll \int_1^x R(z,y)\,dz \underset{\text{hypothesis}}{\ll} x^{3/2}(x+|y|)^{\varepsilon}$$

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- Comparing the two integrals, we get for every $\varepsilon > 0$:

$$x^{\beta_0+1} \ll \left|c\,x^{\rho_0-iy+1}\right| \ll x^{3/2}(x+|y|)^{\varepsilon}$$

  which gives us the desired contradiction.

THANK YOU!!!

References

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