Consecutive sums of two squares in arithmetic progressions.

(joint with Noam Kimmel)

Throughout, fix $q$ modulus and assume $(a,q) = 1$.

Q: Are there infinitely many $p = a \mod q$?

A: Yes. (Dirichlet) Landau, Iwaniec

Stronger: equidistribution

$$\pi(x; a, q) := \# \{ p \leq x : p \equiv a \mod q \}$$

$$\sigma(x; a, q) \sim \frac{\pi(\alpha)}{\phi(q)} \sum_{a, q} \sigma(x)$$

Q: $\pi(x; (a_1, a_2), q) := \# \{ p \leq x : p \equiv a_1 \mod q \}$

$$\sigma(x; (a_1, a_2), q) \rightarrow \infty \quad \text{as} \quad x \rightarrow \infty ?$$

Q: $\pi(x; (a_1, \ldots, a_r), q) := \# \{ p_n \leq x : p_{n+i-1} \equiv a_i \mod q \}$

$$\sigma(x; (a_1, \ldots, a_r), q) \rightarrow \infty ?$$

Conj: $\sim \frac{\pi(\alpha)}{\phi(q)^r} \sum_{a, q} \sigma(x)$

Conj: Due to second-order terms, repeated values are less
Easy Lemma: If \( \varphi(q) = 2 \) and \( a_1 \neq a_2 \mod q \), then \( \pi(x; (a_1, a_2), q) \to \infty \).

Thm (Shiu, Banks–Freiberg–Turnage–Butcherbough): \( \pi(x; (a, \ldots, a), q) \to \infty \) for any length.

Thm (Maynard): \( \pi(x; (a, \ldots, a), q) \gg \pi(x) \).

Open Qn: Show \( \pi(x; \bar{a}, q) \to \infty \) in any other case.

Sums of two squares:

\[
E = \{ 1, 2, 4, 5, 6, 8, 9, 10, 13, \ldots \} = \{ z \in \mathbb{N} : z = x^2 + y^2, x, y \in \mathbb{N} \}
\]

Thm (Fermat): \( n \in E \iff n = \prod_{p \mid n} p^{v_p}, \ \text{where} \ \forall p \equiv 3 \mod 4 \).

\[
E = (E_n) \text{ where } E_n \leq E_{n+1}
\]

Thm (Kimmel, K.): \( \forall a, b, c \mod q \),

\[
\pi(x; (a, b, c), q) \to \infty \ \text{as} \ x \to \infty.
\]

Thm (Kimmel, K.): \( \forall a, b \ \mod q \),

\[
\pi(x; (a, \ldots, a, b, \ldots, b), q) \to \infty \ \text{for any lengths of} \ a's \ \text{and} \ b's.
\]
In progress: \( \sigma(x; (a, \ldots, a, b, \ldots, b), q) \Rightarrow \sigma(x) \).

For triples:

**Thm (Hooley):** \( \forall h, k \in \mathbb{N}, \)

\[ \sum_{n \in \mathbb{N}} \mathcal{I}_E(n) \mathcal{I}_E(n+h) \mathcal{I}_E(n+k) \rightarrow \infty. \]

Note: Techniques from quadratic forms are very helpful for sums of two squares.

**Eg:** infinitely many \( n \) w/ \( n-1, n, n+1 \) \( \in \mathbb{E} \).

**Pf:** if \( n-1, n, n+1 \) is an example

\( n^2 - 1, n^2, n^2 + 1 \) is another

\( (n-1)(n+1) \)

\( 8, 9, 10 \) works.

**Spoiler:** We can estimate certain weighted correlations of \( \mathbb{Z} + \mathbb{I} \).

Back to primes:

**Ideas of proof (BFI/N):**

“admissible”

Maynard: For \( \forall m \in \mathbb{N} \), let \( k \) be big enough. For each \( k \)-tuple

\[ L_1(n) = qn + a_1, \ldots \quad L_k(n) = qn + a_k, \]
\[ \exists \text{ only many } n \text{ s.t. at least } m \text{ of the values } \\
L_1(n), \ldots, L_k(n) \text{ are all prime.} \]

**Trick to get BFT:**

1. Choose \( \{ L_i(n) = q_n + a_i \} \) admissible, with \( a_i \equiv a \mod q \) and \( a_1 < \ldots < a_k \).

2. Define
   
   \[ S = \{ t \in \mathbb{N} : t \neq a_i \; \forall i, \; a_1 < t < a_k \} \]
   
   \[ Q = \prod_{t \in S} q_t \] s.t. \( t \not\equiv a_i \mod q \).

3. **CRT:** \( \exists A \mod Q \) s.t.
   
   \[ qA + t \equiv 0 \mod q_t \quad \forall t. \]

4. \( \{ \widetilde{L}_i(n) = qQn + qA + a_i \} \) is admissible
   
   and \( qQn + qA + t \) is never prime for \( t \in S \).
The output of Maynard's thm for $\tilde{\mathcal{L}}_2(n)$ are consecutive.

$Q = \prod_t q_t^2$

$q_{A+t} \equiv q_t \mod q_t^2$

\[ \equiv a \equiv a \equiv a \equiv a \equiv a \]

\[ \equiv a \equiv a \equiv a \]

\[ \equiv a \equiv b \equiv a \equiv b \equiv a \]

ask for a pink prime and a red prime.

Idea: Divide the tuple into baskets and look for a prime in each basket.

Dream:

\begin{align*}
\text{basket} & \equiv a \mod g \\
\text{basket} & \equiv b \mod g
\end{align*}
\[ L_1(n) \ldots \ldots L_k(n) \quad L_{k+1}(n) \ldots \ldots L_{2k}(n) \]

infinitely often \( \exists \) a prime in each basket.

How to find primes in baskets:

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**Maynard's original idea:**

\[ S = \sum_{n \sim N} \left( \sum_{i=1}^{k} I_p(L_i(n)) - 1 \right) w(n) \]

Find \( w(n) \geq 0 \) s.t. \( S > 0 \) for some \( n \sim N \)

\[ \exists \, n \, s.t. \, \sum_{i=1}^{k} I_p(L_i(n)) > 1. \]

Need to estimate \( \sum_n w(n) \), \( \sum_n I_p(L_i(n)) w(n) \).

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**2nd moment argument:** Divide a \( Bk \)-tuple into \( B \) equal baskets.

\[ S_1 = \sum_{n \sim N} \left( \sum_{i=1}^{Bk} I_p(L_i(n)) - 1 - \sum_{l=1}^{B} \sum_{i,j \in B_l} I_p(L_i(n)) I_p(L_j(n)) \right) w(n) \]

\[ > 0 \]

for some \( w(n) > 0 \).
Here we also need to understand

\[ \sum_{n \sim N} \tau_p(L_i(n)) \tau_p(L_j(n)) w(n) \]

upper bounds lose a factor of \(4\) (or \(\approx 4\))

\underline{Banks - Freiberg - Maynard}: One can find primes in two different baskets if you start with \(\geq 5\) baskets.

\underline{Menkovski}:

\(\geq 4\)

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\underline{McKinnon}: For \(\Omega + \Omega\), you can divide a tuple into \(\Omega\) equal baskets and find infinitely often a \(\Omega + \Omega\) in each basket.