Quantitative upper bounds related to an isogeny criterion for elliptic curves

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Joint with

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Motivation: An Isogeny Criterion for Elliptic Curves

- $E_i, i = 1, 2$: non-CM elliptic curves over a number field K.
- $\mathfrak{p}$: prime of good reduction for E_1 and E_2.
- $\pi_{\mathfrak{p}}(E_i)$, $i = 1, 2$: Frobenius endomorphism of the reduction E_i mod $\mathfrak{p}$.
- $\mathbb{Q}(\pi_{\mathfrak{p}}(E_i))$, $i = 1, 2$: Frobenius fields of E_i at $\mathfrak{p}$.
- $a_{\mathfrak{p}}(E_i)$, $i = 1, 2$: Frobenius trace of E_i at $\mathfrak{p}$, $a_{\mathfrak{p}}(E_i) = \pi_{\mathfrak{p}}(E_i) + \overline{\pi_{\mathfrak{p}}(E_i)}$.

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- **Isogeny criterion**: If $E_1 \sim_{\overline{K}} E_2$, then $\mathbb{Q}(\pi_{\mathfrak{p}}(E_1)) = \mathbb{Q}(\pi_{\mathfrak{p}}(E_2))$ for all but finitely many $\mathfrak{p}$.

Motivation: An Isogeny Criterion for Elliptic Curves

Theorem (Kulkarni-Patankar-Rajan (2016))

$$E_1 \not\sim_{\overline{K}} E_2 \iff \mathcal{F}_{E_1,E_2}(x) = o\left(\frac{x}{\log x}\right),$$

where $\mathcal{F}_{E_1,E_2}(x) := \#\{\mathfrak{p} : \mathrm{N}_K(\mathfrak{p}) \leq x, \mathbb{Q}(\pi_{\mathfrak{p}}(E_1)) = \mathbb{Q}(\pi_{\mathfrak{p}}(E_2))\}$.

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Conjecture (Kulkarni-Patankar-Rajan (2016))

$$E_1 \not\sim_{\overline{K}} E_2 \iff \mathcal{F}_{E_1,E_2}(x) \ll_{E_1,E_2,K} \frac{x^{\frac{1}{2}}}{\log x}.$$

Known Results

Let $E_1/\mathbb{Q}$, $E_2/\mathbb{Q}$ be non-CM, not $\overline{\mathbb{Q}}$-isogenous elliptic curves.

Baier-Patankar (2018)

- Under GRH: $\mathcal{F}_{E_1,E_2}(x) \ll_{E_1,E_2,\epsilon} x^{\frac{29}{30}+\epsilon}$.
- Unconditionally: $\mathcal{F}_{E_1,E_2}(x) \ll_{E_1,E_2} \frac{x(\log\log x)^{\frac{22}{21}}}{(\log x)^{\frac{43}{42}}}$.

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Let E_1/K, E_2/K be non-CM, not $\overline{K}$-isogenous elliptic curves.

A remark of Serre (2005)

- Under GRH:

$$\tilde{\mathcal{F}}_{E_1,E_2}(x) := \#\{\mathfrak{p} : \mathrm{N}_K(\mathfrak{p}) \leq x, \mathbb{Q}(\pi_\mathfrak{p}(E_1)) = \mathbb{Q}(\pi_\mathfrak{p}(E_2)) \notin \{\mathbb{Q}(i), \mathbb{Q}(i\sqrt{3})\}\} \ll x^{\frac{11}{12}}.$$

Motivation: Strong Multiplicity One Theorem

- $\pi_i, i = 1, 2$: unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_K)$, where $\mathbf{A}_K$ is the adéle ring of the number field K.
- $\pi_{i,\mathfrak{p}}, i = 1, 2$: corresponding irreducible local representation of $\mathrm{GL}_2(K_\mathfrak{p})$.
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Jacquet, Piatetski-Shapiro (1979), Shalika (1983)

$\pi_1 \simeq \pi_2 \Leftrightarrow a_\mathfrak{p}(\pi_1) = a_\mathfrak{p}(\pi_2)(\Leftrightarrow \pi_{1,\mathfrak{p}} \simeq \pi_{2,\mathfrak{p}})$ for all but finitely many $\mathfrak{p}$.

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Ramakrishnan (2000)

Let π_1, π_2 be the cuspidal representations associated to **holomorphic modular forms**.

$\mathrm{Ad}(\pi_1) \simeq \mathrm{Ad}(\pi_2) \Leftrightarrow |a_p(\pi_1)| = |a_p(\pi_2)|$ for all but finitely many p ($\Leftrightarrow \pi_1 \simeq \pi_2 \otimes \chi$, χ a Drichlet character).

Motivation: Strong Multiplicity One Theorem

Let π_1, π_2 correspond to **non-CM** newforms of weight $k_1, k_2 \geq 2$ and level q_1, q_2, then

$$\mathrm{Ad}(\pi_1) \not\simeq \mathrm{Ad}(\pi_2) \iff \mathcal{F}_{\pi_1,\pi_2}(x) := \#\{p \leq x : |a_p(\pi_1)| = |a_p(\pi_2)|\} = o\left(\frac{x}{\log x}\right).$$

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Assume $\mathrm{Ad}(\pi_1) \not\simeq \mathrm{Ad}(\pi_2)$.

Murty-Pujahari (2017) Under GRH (for certain Rankin-Selberg L-functions of the symmetric powers of π_1, π_2):

$$\mathcal{F}_{\pi_1,\pi_2}(x) \ll_{k_1,k_2,q_1,q_2} \frac{x^{\frac{7}{8}}}{(\log x)^{\frac{1}{2}}}.$$

Wong (2018)

- Under GRH (same as above):

$$\mathcal{F}_{\pi_1,\pi_2}(x) \ll_{k_1,k_2,q_1,q_2} \frac{x^{\frac{5}{6}}}{(\log x)^{\frac{1}{3}}}.$$

- Unconditionally:

$$\mathcal{F}_{\pi_1,\pi_2}(x) \ll_{k_1,k_2,q_1,q_2,\epsilon} \frac{x}{(\log x)(\log\log x)^{\frac{1}{2}-\epsilon}}.$$

Main Result

Theorem (Cojocaru-Hinz-W., 2024)

Let E_1 and E_2 be non-CM elliptic curves over a number field K, and not $\overline{K}$-isogenous. Let

$$\mathcal{F}_{E_1,E_2}(x) := \#\{\mathfrak{p} : \mathrm{N}_K(\mathfrak{p}) \leq x, \mathbb{Q}(\pi_{\mathfrak{p}}(E_1)) = \mathbb{Q}(\pi_{\mathfrak{p}}(E_2))\}.$$

Then for any sufficiently large x,

- *Unconditionally:*

$$\mathcal{F}_{E_1,E_2}(x) \ll_{E_1,E_2,K} \frac{x(\log\log x)^{\frac{1}{9}}}{(\log x)^{\frac{19}{18}}}.$$

- *Under GRH for Dedekind zeta functions:*

$$\mathcal{F}_{E_1,E_2}(x) \ll_{E_1,E_2,K} \frac{x^{\frac{6}{7}}}{(\log x)^{\frac{5}{7}}}.$$

Proof Strategy

Let α_1 and α_2 be coprime integers. Consider:

$$\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x) := \#\{\mathfrak{p} : \mathrm{N}_K(\mathfrak{p}) \leq x, \alpha_1 a_{\mathfrak{p}}(E_1) + \alpha_2 a_{\mathfrak{p}}(E_2) = 0\}.$$

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$\mathcal{F}_{E_1,E_2}(x) \ll$

$$\mathcal{T}_{E_1,E_2}^{1,1}(x) + \mathcal{T}_{E_1,E_2}^{1,-1}(x) + \sum_{1\leq i\leq 2} \#\{\mathfrak{p} : \mathrm{N}_K(\mathfrak{p}) \leq x, \mathbb{Q}(\pi_{\mathfrak{p}}(E_i)) \in \{\mathbb{Q}(i), \mathbb{Q}(i\sqrt{3})\}\}.$$

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$\#\{\mathfrak{p} : \mathrm{N}_K(\mathfrak{p}) \leq x, \mathbb{Q}(\pi_{\mathfrak{p}}(E_i)) \in \{\mathbb{Q}(i), \mathbb{Q}(i\sqrt{3})\}\}$ is known (Zywina (2015)):

- Unconditionally: $\ll_{E_i,K} \frac{x(\log\log x)^2}{(\log x)^2}$
- Under GRH: $\ll_{E_i,K} \frac{x^{\frac{4}{5}}}{(\log x)^{\frac{3}{5}}}$

Related to the Lang-Trotter Conjecture for Frobenius fields of non-CM elliptic curves.

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$\Longrightarrow$ **It suffices to estimate** $\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x)$.

Estimation of $\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x)$: Galois representation

Let N_{E_1}, N_{E_2} be the conductors of E_1, E_2.

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Consider modulo ℓ representation of $E_1 \times E_2$, for some $\ell = \ell(x)$:

$$\overline{\rho}_{E_1 \times E_2,\ell} = (\overline{\rho}_{E_1,\ell}, \overline{\rho}_{E_2,\ell}) : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E_1 \times E_2[\ell]) \simeq \mathrm{GL}_2(\mathbb{F}_\ell) \times \mathrm{GL}_2(\mathbb{F}_\ell).$$

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Open Image Theorem for $E_1 \times E_2$: for $\ell \gg 1$,

$\mathrm{Gal}(K_\ell/K) \simeq \mathrm{Im}(\overline{\rho}_{E_1\times E_2,\ell}) = G(\ell) := \{(M_1, M_2) \in \mathrm{GL}_2(\mathbb{F}_\ell) \times \mathrm{GL}_2(\mathbb{F}_\ell) : \det M_1 = \det M_2\}.$

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$\alpha_1 \operatorname{tr}(\overline{\rho}_{E_1,\ell}(\mathrm{Frob}_\mathfrak{p})) + \alpha_2 \operatorname{tr}(\overline{\rho}_{E_2,\ell}(\mathrm{Frob}_\mathfrak{p})) \equiv \alpha_1 a_\mathfrak{p}(E_1) + \alpha_2 a_\mathfrak{p}(E_2) \equiv 0 \pmod{\ell}$, $\mathfrak{p} \nmid \ell N_{E_1} N_{E_2}$.

$\mathcal{C}_0(\ell)^{\alpha_1,\alpha_2} := \{(M_1, M_2) \in G(\ell) : \alpha_1 \operatorname{tr} M_1 + \alpha_2 \operatorname{tr} M_2 = 0\}$

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$$\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x) \leftrightarrow \pi_{\mathcal{C}_0(\ell)^{\alpha_1,\alpha_2}}(x, K_\ell/K).$$

Variations of the Chebotarev Counting Functions

$$B(\ell) := \left\{ \left(\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right) \in G(\ell) \right\}, U'(\ell) := \left\{ \left(\begin{pmatrix} a & * \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & * \\ 0 & a \end{pmatrix} \right) \in G(\ell), a \in \mathbb{F}_\ell^\times \right\},$$

$$\Lambda(\ell) := \left\{ (aI, aI) \in G(\ell) : a \in \mathbb{F}_\ell^\times \right\}, P(\ell) := G(\ell)/\Lambda(\ell).$$

$$\mathcal{C}(\ell)^{\alpha_1,\alpha_2} := \left\{ (M_1, M_2) \in \mathcal{C}_0(\ell)^{\alpha_1,\alpha_2} : \lambda_1(M_j), \lambda_2(M_j) \in \mathbb{F}_\ell^\times \ \forall 1 \le j \le 2 \right\},$$

$\widehat{\mathcal{C}}_{\text{Borel}}(\ell)^{\alpha_1,\alpha_2} :=$ the image of $\mathcal{C}(\ell)^{\alpha_1,\alpha_2} \cap B(\ell)$ in $B(\ell)/U'(\ell)$ (abelian).

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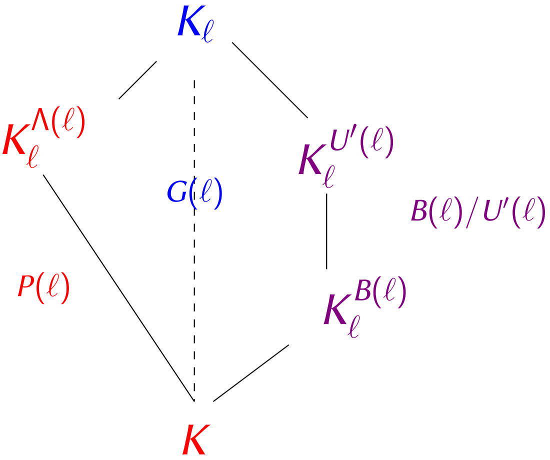
- Under GRH: $\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x) \leftrightarrow \max\limits_{y(x) \le \ell \le y(x)+u(x)} \pi_{\mathcal{C}(\ell)^{\alpha_1,\alpha_2}}(x, K_\ell/K)$

$$\leftrightarrow \max_{y(x) \le \ell \le y(x)+u(x)} \pi_{\widehat{\mathcal{C}}_{\text{Borel}}(\ell)^{\alpha_1,\alpha_2}}\left(x, K_\ell^{U'(\ell)}/K_\ell^{B(\ell)}\right).$$

Field Exrensions



Uncnditional estimation of $\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x)$

By unconditionally effective Chebotarev Density Theorem:

$$\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x) \ll \pi_{\widehat{\mathcal{C}}_{\mathrm{Proj}}(\ell)^{\alpha_1,\alpha_2}}\left(x, K_\ell^{\Lambda(\ell)}/K\right) \ll \frac{|\widehat{\mathcal{C}}_{\mathrm{Proj}}(\ell)^{\alpha_1,\alpha_2}|}{|P(\ell)|}\,\mathrm{li}(x) \ll \frac{x}{\ell \log x},$$

as long as

$$\log x \gg n_K^3 \ell^{18} (\log(\ell N_{E_1} N_{E_2} d_K))^2 \left(\gg [K_\ell^{\Lambda(\ell)} : K] \left(\log |d_{K_\ell^{\Lambda(\ell)}}|\right)^2\right).$$

Uncnditional estimation of $\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x)$

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as long as

$$\log x \gg n_K^3 \ell^{18} (\log(\ell N_{E_1} N_{E_2} d_K))^2 \left(\gg [K_\ell^{\Lambda(\ell)} : K] \left(\log |d_{K_\ell^{\Lambda(\ell)}}|\right)^2 \right).$$

We take

$$\ell(x) = \left[a \frac{(\log x)^{\frac{1}{18}}}{(\log\log x)^{\frac{1}{9}}} \right]$$

for some positive constant $a = a(h_A, n_K, d_K, N_{E_1}, N_{E_2}, \alpha_1, \alpha_2)$, we get

$$\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x) \ll_{E_1,E_2,K,\alpha_1,\alpha_2} \frac{x(\log\log x)^{\frac{1}{9}}}{(\log x)^{\frac{19}{18}}}.$$

Estimation of $\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x)$ under GRH

$$\mathcal{T}_{E_1,E_2}^{\alpha_1,\alpha_2}(x) \ll \max_{y(x)\leq \ell \leq y(x)+u(x)} \pi_{\widehat{\mathcal{C}}_{\mathrm{Borel}(\ell)^{\alpha_1,\alpha_2}}}\left(x, K_\ell^{U'(\ell)}/K_\ell^{B(\ell)}\right).$$

Estimation of $\mathcal{T}^{\alpha_1,\alpha_2}_{E_1,E_2}(x)$ under GRH

$$\mathcal{T}^{\alpha_1,\alpha_2}_{E_1,E_2}(x) \ll \max_{y(x)\le\ell\le y(x)+u(x)} \pi_{\widehat{\mathcal{C}}_{\mathrm{Borel}(\ell)^{\alpha_1,\alpha_2}}}\left(x, K_\ell^{U'(\ell)}/K_\ell^{B(\ell)}\right).$$

Effective Chebotarev Density Theorem under GRH and AHC:

$$\begin{aligned}\pi_{\mathcal{C}(\ell)^{\alpha_1,\alpha_2}}(x, K_\ell/K) \quad &\ll \quad \frac{\left|\widehat{\mathcal{C}}_{\mathrm{Borel}}(\ell)^{\alpha_1,\alpha_2}\right| \cdot |U'(\ell)|}{|B(\ell)|} \cdot \frac{x}{\log x} \\ &+ \quad \left|\widehat{\mathcal{C}}_{\mathrm{Borel}}(\ell)^{\alpha_1,\alpha_2}\right|^{\frac{1}{2}} [K_\ell^{B(\ell)} : K] \frac{x^{\frac{1}{2}}}{\log x} \log M\left(K_\ell^{U'(\ell)}/K_\ell^{B(\ell)}\right) \\ &\ll \quad \frac{x}{\ell \log x} + \ell^{\frac{5}{2}} \frac{x^{\frac{1}{2}}}{\log x} \cdot \frac{\log(\ell N_1 N_2 d_K)}{n_K}.\end{aligned}$$

$$y(x) = \left[a' \frac{x^{\frac{1}{7}}}{(\log x)^{\frac{2}{7}}}\right], \quad u(x) = \left[a'' y(x)^{\frac{1}{2}} (\log y(x))^{2+\varepsilon}\right]$$

for some positive constants $a' = a'(h_A, n_K, d_K, N_1, N_2, \alpha_1, \alpha_2)$ and $a'' = a''(h_A, n_K, d_K, N_1, N_2, \alpha_1, \alpha_2)$ and get

$$\mathcal{T}^{\alpha_1,\alpha_2}_{E_1,E_2}(x) \ll_{E_1,E_2,K,\alpha_1,\alpha_2} \frac{x^{\frac{6}{7}}}{(\log x)^{\frac{5}{7}}}.$$

THANKS FOR YOUR ATTENTION!