A Weyl-type inequality for irreducible elements in function fields, with applications

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Weyl differencing

Let us begin with the differencing process. Write \( e(x) = e^{2\pi i x} \) for real \( x \). Let \( f(x) = \sum_{j=0}^{k} \alpha_j x^j \in \mathbb{R}[x] \). Weyl observed that

\[
\left| \sum_{n=1}^{N} e(f(n)) \right|^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} e(f(m) - f(n))
\]

\[
= N + 2\text{Re} \sum_{\ell=1}^{N-1} \sum_{n=1}^{N-\ell} e(f(n + \ell) - f(n)).
\]

Note that \( f(n + \ell) - f(n) = g_\ell(n) \) is a polynomial of degree \( k - 1 \).

This process is known as **Weyl differencing**.

One can continue the process \( k - 1 \) times and reduce the exponent to a linear polynomial.
In $\mathbb{R}$, a sequence $(a_n)_{n=1}^{\infty}$ of real numbers is **equidistributed** (mod 1) if for any interval $I \subset [0, 1)$, we have

$$\lim_{N \to \infty} \frac{\# \{a_n : 1 \leq n \leq N \text{ and } \{a_n\} \in I\}}{N} = |I|,$$

where $\{a\}$ is the fractional part of $a$. Using the differencing process, Weyl proved the classical equidistribution theorem.

**Theorem (Weyl, 1916)**

If $f(x)$ is a polynomial with real coefficients and at least one of the non-constant coefficients is irrational, then the sequence $\{f(n)\}$ is equidistributed (mod 1).

In the same paper, using the idea of differencing, Weyl also proved the famous inequality (Weyl's inequality), although it was given in a less explicit form.
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In the same paper, using the idea of differencing, Weyl also proved the famous inequality (Weyl’s ineq), although it was given in a less explicit form.
Theorem (Weyl’s inequality, an explicit form)

Suppose that \( f(x) = \sum_{j=0}^{k} \alpha_j x^j \in \mathbb{R}[x] \), and that \( |\alpha_k - a/q| < q^{-2} \), \((a, q) = 1\). Then for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{N} e(f(n)) \ll_{k, \varepsilon} N^{1+\varepsilon} \left( \frac{1}{q} + \frac{1}{N} + \frac{q}{N^k} \right)^{2^{1-k}}.
\]

Theorem (Weyl’s inequality, an inverse form)

Given \( 0 < \eta \leq 2^{1-k} \), for any \( \varepsilon > 0 \), if \( N \) is sufficiently large in terms of \( \varepsilon \) and \( \eta \), and

\[
\left| \sum_{n=1}^{N} e(f(n)) \right| > N^{1-\eta},
\]

then there are \((a, q) = 1\), such that

\[
q < Z_{\eta, \varepsilon, k} = N^{\varepsilon+2^{k-1}\eta} \quad \text{and} \quad |q\alpha_k - a| < Z_{\eta, \varepsilon, k}/N^k.
\]
Weyl’s inequality over primes in $\mathbb{Z}$

**Theorem (Harman)**

Suppose that $f(x) = \sum_{j=0}^{k} \alpha_j x^j \in \mathbb{R}[x]$, and that $|\alpha_k - a/q| < q^{-2}$, $(a, q) = 1$. Then for any $\varepsilon > 0$,

$$\sum_{p \leq N} (\log p) e(f(p)) \ll_{k, \varepsilon} N^{1+\varepsilon} \left( \frac{1}{q} + \frac{1}{N^{1/2}} + \frac{q}{N^k} \right)^{4^{1-k}}.$$

Weyl-type inequality in $\mathbb{F}_q[t]$
As a key ingredient in the Hardy-Littlewood Method, the Weyl-type inequality is applied in many problems.

- Waring’s problem, Goldbach’s problem...
- Diophantine inequalities, Diophantine equations...
- Sumsets problems, Sequences...
- Riemann zeta-function, $L$-functions...
Ring of polynomials over $\mathbb{F}_q$

Let $\mathbb{F}_q[t]$ be the polynomial ring over a finite field with $q$ elements and characteristic $p$.

Let

$$K = \mathbb{F}_q(t) = \left\{ \frac{x}{y} : x, y \in \mathbb{F}_q[t], y \neq 0 \right\}$$

be the field of fractions, and let

$$K_\infty = \mathbb{F}_q((1/t)) = \left\{ \sum_{j=-\infty}^{N} a_j t^j : a_j \in \mathbb{F}_q, N \in \mathbb{Z} \right\}.$$

For $\alpha = \sum_{j=-\infty}^{N} a_j t^j \in K_\infty$ with $a_N \neq 0$, we define $\text{ord}(\alpha) = N$ and $|\alpha| = q^{\text{ord}\alpha}$. In particular, $\text{ord}(0) = -\infty$. 
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Here, $\mathbb{F}_q[t]$, $K$, $K_\infty$ play the roles of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.
Exponential function on $K_\infty$

Define \( \{\alpha\} = \sum_{j=-\infty}^{-1} a_j t^j \) to be the \textbf{fractional part} of \( \alpha \) and let \( \text{res}(\alpha) = a_{-1} \). Then,

\[
T = \left\{ \sum_{j=-\infty}^{-1} a_j t^j : a_j \in \mathbb{F}_q \right\}
\]

is the analog of \([0, 1)\) in \(\mathbb{R}\).
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is the analog of $[0, 1)$ in $\mathbb{R}$.

Let $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ denote the trace map. Then for $\alpha \in K_\infty$, the exponential function is defined as

$$e(\alpha) := e^{2\pi i \cdot \text{tr}(\text{res}\alpha)/p}.$$ 

This is an additive character on $K_\infty$ and analogous to $e^{2\pi ix}$ in $\mathbb{R}$. We can use this function to study additive problems in function fields.
Weyl differencing is problematic in $\mathbb{F}_q[t]$.

Q: Can we use the differencing process to prove an analog of Weyl’s inequality?
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**Q:** Can we use the differencing process to prove an analog of Weyl’s inequality?

Let $f(x) = \sum_{j=1}^{k} \alpha_j x^j$, $\alpha_j \in \mathbb{K}_\infty$.

- If $k < p = \text{char}(\mathbb{F}_q)$, then one can repeat Weyl differencing and prove analogous results.

- If $k \geq p$, Weyl differencing is problematic. Look at the leading coefficient of $f(x)$. If we do $f(x + h) - f(x)$, $k - 1$ times, we end up having a factor of $k!$ in the final leading coefficient, which is 0 when $k \geq p$.  

Y.-R. Liu and T. Wooley (2010), in their Waring's problem paper, overcame the barrier of $k < p$ in function fields, by using large sieve and Vinogradov’s mean value theorem (VMVT).
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Carlitz’s Example

For any $x = \sum_{j=0}^{n} c_j t^j \in \mathbb{F}_q[t]$, we have $x^p = \sum_{j=0}^{n} c_j^p t^{jp} \in \mathbb{F}_q[t^p]$.

**Example.** (Carlitz, 1952) Let

$$C = \left\{ \alpha : \alpha = \sum_{i=-\infty}^{n} c_i t^i, c_{-jp-1} = 0 \text{ for all } j \right\},$$

so that $\exp(\alpha x^p) = 1$ for all $x \in \mathbb{F}_q[t]$. 

Weyl-type inequality in $\mathbb{F}_q[t]$
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**Example.** (Carlitz, 1952) Let

$$C = \left\{ \alpha : \alpha = \sum_{i=-\infty}^{n} c_i t^i, c_{jp-1} = 0 \text{ for all } j \right\},$$

so that $e(\alpha x^p) = 1$ for all $x \in \mathbb{F}_q[t]$.

Weyl-type inequality: if $| \sum e(\alpha x^p) |$ is large, can the leading coefficient $\alpha$ be well-approximated by rationals with small denominators?

There are many (irrational) $\alpha \in C$ that cannot be well-approximated by rationals.
Carlitz’s Example

For any \( x = \sum_{j=0}^{n} c_j t^j \in \mathbb{F}_q[t] \), we have \( x^p = \sum_{j=0}^{n} c_j^p t^{jp} \in \mathbb{F}_q[t^p] \).

Example. (Carlitz, 1952) Let

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\mathcal{C} = \left\{ \alpha : \alpha = \sum_{i=-\infty}^{n} c_i t^i, c_{jp-1} = 0 \text{ for all } j \right\},
\]

so that \( e(\alpha x^p) = 1 \) for all \( x \in \mathbb{F}_q[t] \).

Weyl-type inequality: if \( |\sum e(\alpha x^p)| \) is large, can the leading coefficient \( \alpha \) be well-approximated by rationals with small denominators? There are many (irrational) \( \alpha \in \mathcal{C} \) that cannot be well-approximated by rationals.

Example. For polynomials like \( f(x) = \alpha x^p + \beta x \), it is not possible to determine the Diophantine approximation of \( \alpha \) or \( \beta \) by the Weyl sum, since \( x^p \) and \( x \) interfere with one another.
Q: Given \( f(x) = \sum_{j \in \mathcal{K}} \alpha_j x^j \in K_\infty[x] \) supported on \( \mathcal{K} \subset \mathbb{Z}^+ \), which coefficients satisfy Weyl-type inequalities?

**Example**

Suppose \( p = 7 \) and \( \mathcal{K} = ([1, 3p + 1] \cap \mathbb{Z}) \cup \{p^3 + p^2, 3p^4, p^6 + 2p^5\} \).

To visualize it, we plot \( \mathcal{K} \) on the number line in the following way.
Q: Given $f(x) = \sum_{j \in \mathcal{K}} \alpha_j x^j \in \mathbb{K}_\infty[x]$ supported on $\mathcal{K} \subset \mathbb{Z}^+$, which coefficients satisfy Weyl-type inequalities?

Example

Suppose $p = 7$ and $\mathcal{K} = ([1, 3p + 1] \cap \mathbb{Z}) \cup \{p^3 + p^2, 3p^4, p^6 + 2p^5\}$.

To visualize it, we plot $\mathcal{K}$ on the number line in the following way.

Ideally, the set of indices (in green) without interference is the largest subset of $\mathcal{K}$ on which Weyl’s inequality applies.
Given a finite set $\mathcal{K} \subset \mathbb{Z}^+$, define the set (without interference)

$$I_{\mathcal{K}} = \{ k \in \mathcal{K} : p \nmid k, kp^v \notin \mathcal{K} \text{ for any positive integer } v \}.$$
Given a finite set \( \mathcal{K} \subset \mathbb{Z}^+ \), define the set (without interference)

\[
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\]

1. **Define the shadow of** \( \mathcal{K} \) **to be**

   \[
   S(\mathcal{K}) := \{ j \in \mathbb{Z}^+ : p \nmid \binom{r}{j} \text{ for some } r \in \mathcal{K} \}.
   \]

2. **Define** \( \mathcal{K}^* := \{ k \in \mathcal{K} : p \nmid k \text{ and } p^v k \notin S(\mathcal{K}) \text{ for any } v \in \mathbb{Z}^+ \} \) **to “remove” interfering coefficients (indices) on the shadow.**

3. **For** \( \mathcal{K}_0 = \mathcal{K} \), \( \mathcal{K}_n = \mathcal{K}_{n-1} \setminus \mathcal{K}_{n-1}^* \), we define \( \widetilde{\mathcal{K}} := \bigcup_{n \geq 0} \mathcal{K}_n^* \).

Lê-Liu-Wooley proved a Weyl-type inequality for all coefficients \( \alpha_j \) with \( j \in \widetilde{\mathcal{K}} \).

Note that

\[
\widetilde{\mathcal{K}} \subset \mathcal{I}_\mathcal{K} \subset (\mathcal{K} \setminus p\mathbb{Z})
\]
Theorem (Lê-Liu-Wooley, 2023)

Fix \( q \) and a finite set \( K \subset \mathbb{Z}^+ \). There exist positive constant \( c \) and \( C \) depending only on \( K \) and \( q \), such that following holds. Let \( \epsilon > 0 \) and \( N \) sufficiently large (in terms of \( K, \epsilon, q \)). Let \( f(x) = \sum_{r \in K} \alpha_r x^r \in \mathbb{K}_\infty[x] \). If

\[
\left| \sum_{\deg x < N} e(f(x)) \right| \geq q^{N-\eta},
\]

for some \( \eta \in (0, cN] \). Then for each \( k \in \tilde{K} \) there exist \( a \in \mathbb{F}_q[t] \) and monic \( g \in \mathbb{F}_q[t] \) such that

\[
|g \alpha_k - a| < \frac{q^{\epsilon N+C\eta}}{q^{kN}} \quad \text{and} \quad |g| \leq q^{\epsilon N+C\eta}.
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Theorem (Lê-Liu-Wooley, 2023)

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\]

- \( f(x) = \alpha_k x^k + \cdots \) with \( (k, p) = 1 \).
- \( f(x) = \alpha_\ell x^\ell + \cdots + \alpha_k x^k + \cdots \), with \( (k, p) = 1 \) and \( k > \ell/p \).
- \( f(x) = \sum_{1 \leq j \leq k, (j, p) = 1} \alpha_j x^j \). In this case, \( \widetilde{\mathcal{K}} = \mathcal{I} = \mathcal{K} \).
Define the von Mangoldt function over $\mathbb{F}_q[t]$ by $\Lambda(x) = \deg(P)$, if $x = cP^r$ for some monic irreducible $P$, zero otherwise.
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**Theorem (Champagne-G.-Lê-Liu, 2023+)**

Let $\mathcal{K} \subset \mathbb{Z}^+$ be a finite set and $k \in \mathcal{I}_\mathcal{K}$. There exist constants $c_k, C_k > 0$ (depending on $k, \mathcal{K}, q$) such that the following holds:

Let $\epsilon > 0$ and $N$ be sufficiently large in terms of $\mathcal{K}, \epsilon$ and $q$. Suppose that $f(u) = \sum_{r \in \mathcal{K} \cup \{0\}} \alpha_r u^r \in \mathcal{K}_\infty[u]$ satisfying the bound

$$\left| \sum_{x \in \mathcal{A}_N} \Lambda(x)e(f(x)) \right| \geq q^{N-\eta},$$

for some $\eta$ with $0 < \eta \leq c_k N$. Then, there exist $a_k \in \mathbb{F}_q[t]$ and monic $g_k \in \mathbb{F}_q[t]$ such that

$$|g_k \alpha_k - a_k| < \frac{q^{\epsilon N + C_k \eta}}{q^{kN}}$$

and

$$|g_k| \leq q^{\epsilon N + C_k \eta}.$$
Like Weyl proved the equidistribution theorem, Lê-Liu-Wooley (in the same paper) proved the next theorem.

**Theorem (Lê-Liu-Wooley, 2023)**

Let \( f(u) = \sum_{r \in K \cup \{0\}} \alpha_r u^r \) be a polynomial supported on \( K \subset \mathbb{Z}^+ \) with coefficients in \( K_\infty \). Suppose \( \alpha_k \) is irrational for some \( k \in \widetilde{K} \). Then the sequence \( (f(x))_{x \in \mathbb{F}_q[t]} \) is equidistributed in \( \mathbb{T} \).

**Remarks:**

- **Carlitz** (1952) gave a family of irrational \( \alpha \) that \( e(\alpha x^p) = 1 \) for all \( x \in \mathbb{F}_q[t] \), thus equidistribution does not hold for \( f(x) = \alpha x^p \).

\[ \mathbb{P} = \{ x \in F_q[t] : \text{monic irreducible} \} \]
\( \mathbb{P} = \{ x \in \mathbb{F}_q[t] : \text{monic irreducible} \} \).

**Theorem (Champagne-G.-Lê-Liu, 2023+)**

Let \( f(u) = \sum_{r \in \mathcal{K} \cup \{0\}} \alpha_r u^r \) be a polynomial supported on \( \mathcal{K} \subset \mathbb{Z}^+ \) with coefficients in \( \mathbb{K}_\infty \). Suppose \( \alpha_k \) is irrational for some \( k \in \mathcal{I}_\mathcal{K} \). Then the sequence \( (f(x))_{x \in \mathbb{F}_q[t]} \) is equidistributed in \( \mathbb{T} \).

- **Carlitz** (1952): the result may not hold for \( f(x) = \alpha x^p \).
- **Rhin** (1972) proved the theorem when \( \mathcal{K} = \{1\} \).
- **Difficulty**: The space \( \mathbb{P} \) is not self-similar as \( \mathbb{F}_q[t] \). A Weyl-type inequality does not immediately imply the equidistribution theorem.
  1. We prove for the special case \( \tilde{\mathcal{K}} = \mathcal{I}_\mathcal{K} = \mathcal{K} \), for which we further prove an epsilon-free version of Weyl’s inequality.
  2. Then we prove the equidistribution theorem on \( \mathcal{I}_\mathcal{K} \) for general \( \mathcal{K} \), using Jérémy Champagne’s argument.
Application 2: Additive inequality of irreducible powers

Let \( \mathbb{P}^k_{kN} = \{x^k : x \text{ is monic irreducible, } \deg(x^k) = kN\} \).

**Theorem (G.)**

Suppose \((p, k) = 1\) and \(k \geq 2\). Let \(N\) be a large number. Let \( \mathcal{A} \) be a set of polynomials in \( \mathbb{F}_q[t] \) of degree less than \( kN \) and \( 0 < \frac{|\mathcal{A}|}{q^{kN}} = \delta < e^{-2} \). Then we have

\[
\frac{|\mathcal{A} + \mathbb{P}^k_{kN}|}{q^{kN}} > \delta \frac{4 \log(2) + c_q \log(k)}{\log \log(1/\delta)}
\]

for some \( c_q > 0 \).

- It is different from the analog in \( \mathbb{Z} \) that the theorem is not true when \( p \mid k \).
- Among all monic degree-\( kN \) polynomials, the proportion (density) of \( \mathbb{P}^k_{kN} \) is very tiny. However, \( \mathcal{A} + \mathbb{P}^k_{kN} \) is significantly denser than \( \mathcal{A} \) for every small density set \( \mathcal{A} \).
Ingredients of Lê-Liu-Wooley’s original method include

- Weyl’s shift,
- Large sieve inequality (Hsu),
- Vinogradov’s mean value theorem (Liu-Wooley).
Ingredients of the Proof

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More tools for irreducible elements:

- Vaughan’s identity in $\mathbb{F}_q[t]$.
- A bootstrap argument. (Iterate LLW’s argument multiple times.)
- Major arc estimates for removing the epsilon.
- A nice self-duality property of $\mathbb{K}_\infty$. 
To help sketch the arguments, we introduce the following notation:

\[ G_N := \{ x \in \mathbb{F}_q[t] : \deg(x) < N \}. \]

This is the analog of \([0, N)\) in integers.
To help sketch the arguments, we introduce the following notation:

\[ \mathbb{G}_N := \{ x \in \mathbb{F}_q[t] : \deg(x) < N \}. \]

This is the analog of \([0, N)\) in integers.

Moreover,

\[ \mathbb{A}_N := \{ x \in \mathbb{F}_q[t] : \text{monic } \deg(x) = N \}. \]

This is the analog of the dyadic interval \([N, 2N)\) in integers.
Lemma (Weyl’s shift)

Let $\mathcal{A} \subset \mathbb{F}_q[t]$ be a multiset consisting of elements of degree less than $N$. We have

$$\sum_{x \in \mathcal{A}_N} e(f(x)) = \#(\mathcal{A})^{-1} \sum_{x \in \mathcal{A}_N} \sum_{y \in \mathcal{A}} e(f(y + x))$$

Proof.

For each $y$ with $\deg(y) < N$, we have

$$\sum_{x \in \mathcal{A}_N} e(f(x)) = \sum_{x \in \mathcal{A}_N} e(f(x + y)).$$

Summing $y \in \mathcal{A}$, the lemma follows.

• The choice of $\mathcal{A}$ is very flexible!

• Instead of looking at a sum over $\mathcal{A}_N$, we turn attention on summing $e(g(x)) = e(f(x + y))$ over $y \in \mathcal{A}$.

• The new polynomial $g(x)$ is supported on the shadow. (Bad)
Lemma (Weyl’s shift)

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Proof. For each $y$ with $\text{deg}(y) < N$, we have

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Summing $y \in A$, the lemma follows.

- The choice of $A$ is very flexible!
- Instead of looking at a sum over $\mathbb{A}_N$, we turn attention on summing $e(g_x(y)) = e(f(x + y))$ over $y \in A$.
- The new polynomial $g_x(y)$ is supported on the shadow. (Bad)
1 Based on Dirichlet’s approximation, we take a multiset \( \mathcal{A} = \{\ell u\} \) that “fit” the approximation and (Weyl) shift the sum onto \( \mathcal{A} \).
   • This turns the original sum into a bilinear sum.
   • It creates well-spaced (leading) coefficients \( \{\alpha \ell^k\} \), i.e. distinct elements are at least \( q^{-\lambda} \) apart in \( \mathbb{T} \) for some \( \lambda > 0 \) (depending on the Diophantine approximation of \( \alpha \)).

2 Then, we apply Hölder’s inequality and Hsu’s large sieve inequality to convert the bilinear sum into Vinogradov’s mean value problem.

3 Finally, we apply Liu-Wooley’s VMVT. The final upper estimate depends on \( q^\lambda \) (and hence the Diophantine approximation of \( \alpha \)).
Define the mobius function $\mu(x) = (-1)^r$ if $x$ is square-free with $r$ distinct monic irreducible factors, zero otherwise.
Vaughan’s identity

Define the mobius function $\mu(x) = (-1)^r$ if $x$ is square-free with $r$ distinct monic irreducible factors, zero otherwise.

Let $1 \leq U, V \leq N$. For every monic $x \in \mathbb{F}_q[t]$ with $\deg(x) < U$, we have

$$\Lambda(x) = a_1(x) + a_2(x) + a_3(x),$$

where

$$a_1(x) = -\sum_{uvw=x \atop u \in G_U \atop v \in G_V} \Lambda(u)\mu(v), \quad a_2(x) = \sum_{uv=x \atop u \in G_V} \deg(u)\mu(v),$$

$$a_3(x) = \sum_{uvw=x \atop \deg(u) \geq U \atop \deg(v) \geq V} \Lambda(u)\mu(v),$$

and the sums are over monic polynomials.
By Vaughan’s identity,

\[ S(N, f) = \sum_{x \in \mathbb{A}_N} \Lambda(x)e(f(x)) = S_1 + S_2 + S_3. \]
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- **Type I sums:**

  \[ J_1 = \sum_{u \in \mathbb{A}_L} \sum_{v \in \mathbb{A}_{N-L}} \phi(u) e(f(uv)). \]

  \[ S_1 \text{ and } S_2 \text{ can be decomposed as linear combination of Type I sums. In particular, when } L = 0, \text{ this is an ordinary exponential sum.} \]
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  \( S_1 \) and \( S_2 \) can be decomposed as linear combination of Type I sums. In particular, when \( L = 0 \), this is an ordinary exponential sum.

- **Type II sums:**
  \[ J_2 = \sum_{u \in \mathbb{P}_L} \sum_{v \in \mathbb{G}_{N-L}} \psi(v)e(f(uv)), \]

  where \( \mathbb{P}_L \) is the set of monic irreducible polynomials of degree \( L \). Using triangle inequality, \( S_3 \) can be bounded by Type II sums.
Le-Liu-Wooley estimated the ordinary exponential sum:

When $(k, p) = 1$ and $|P_{x \in G^N} f(x)| > q^{N - M}$ for some $M$, find a rational approximation: $|b| < q^M$ and $|b\alpha - a| < q^{N - kN + M}$.

In our proof, we consider the problem for the bilinear sums.

- **Type I sums**
  
  $J_1 = \sum_{u \in A^L} \phi(u) \sum_{v \in A^{N-L}} \le(f(uv))$, for $0 \le L \le N - 2M$.

- **Type II sums**
  
  $J_2 = \sum_{u \in P^L} \psi(v) e(f(uv))$, for $0 \le L \le N/2$.

The difficulty is to obtain the same quality of the rational approximation of $\alpha k$ simultaneously for all (large) $L$ in the red range.
Le-Liu-Wooley estimated the ordinary exponential sum:

- When \((k, p) = 1\) and \(|\sum_{x \in \mathbb{G}_N} e(f(x))| > q^{N-M}\) for some \(M\), find a rational approximation: • \(|b| < q^M\) and • \(|b\alpha - a| < q^{-kN+M}\).
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The difficulty is the to obtain the same quality of the rational approximation of \(\alpha_k\) simultaneously for all (large) \(L\) in the red range.
Consider

\[ J_2 = \sum_{u \in P_L} \sum_{v \in G_{N-L}} \psi(v)e(f(uv)). \]

- One can partition \( P_L = \bigcup_i A_i \) (Very flexible)
- After triangle inequality, to study \( J_2 \), it suffices to look at the sum over \( A \):

\[ \sum_{u \in A \subset P_L} \sum_{v \in G_{N-L}} \psi(v)e(f(uv)). \]

These two bullet points are parallel to Weyl’s shift.
• We begin with Dirichlet’s theorem. Accordingly, we pick a family of sets $A$ that “fit” the trivial approximation:

$$|J_2| \leq \sum_i \left| \sum_{u \in A_i} \sum_{v \in \mathbb{G}_{N-L}} \psi(v)e(f(uv)) \right|.$$

• After Holder’s inequality, Hsu’s large sieve, and Liu-Wooley’s theorem, we end up having

If $|J_2| > Tq^{N-M}$ where $|\psi| \leq T$, then there are $(a, b) = 1$ with

$$|b\alpha - a| < q^{-kN+L}, \quad |b| < q^M. \quad (1)$$

The approximation (1) is worse than what we want when $L > M$, but this is still much better than the trivial approximation.

**Remark.** The process in the second bullet point is independent of what $A$ is.
Bootstrap the quality of the approximation

\[ |J_2| \leq \sum_i \left| \sum_{u \in A_i} \sum_{v \in G_{N-L}} \psi(v)e(f(uv)) \right| \]

Next, we repeat LLW’s argument again.

- Suppose \(|J_2| > Tq^{N-M}\). Then we have approximation (1) in hand, which is much better than the trivial approximation.

- Next, we find a new family of \(A\)s that “fit” the approximation (1). We are going to do LLW’s process over this new family of \(A\).

- After Holder’s inequality, Hsu’s large sieve, and Liu-Wooley’s theorem, we end up having:

If \(|J_2| > Tq^{N-M}\) then there are \((a, b) = 1\) with

\[
|b\alpha - a| < q^{-kN+M}, \quad |b| < q^M.
\] (2)
Further remarks

- For \( J_2 = \sum_{u \in P_L} \sum_{v \in G_{N-L}} \psi(v) e(f(uv)) \), we can do \( M \leq L \leq N/2 \) at this moment.

  The barrier \( N/2 \) can be relaxed to \( N \) if one applies Vaughan’s identity to the bilinear sum and repeats the whole process again.

- In the classical Vaughan/Vinogradov’s Type I/II method, type II is usually the more difficult one, but in our case, Type II is the easier one.
Generalizing $\tilde{\mathcal{K}}$ to $\mathcal{I}$

Lemma (Self-duality)

For any $v \in \mathbb{Z}^+ \cup \{0\}$ and $\alpha \in K_\infty$, there exists $\tau = \tau_v(\alpha) \in K_\infty$ such that

$$e(\alpha x^{rp^v}) = e(\alpha(x^r)^{p^v}) = e(\tau x^r)$$

Given a finite $\mathcal{K} \subset \mathbb{Z}^+$, $\mathcal{R} = \mathcal{R}_\mathcal{K} = \{ r : p \nmid r, rp^v \in \mathcal{K} \text{ for some integer } v \}$. Using the above lemma, we can simplify the sum as

$$\sum_x e\left( \sum_{j \in \mathcal{K}} \alpha_j x^j \right) = \sum_x e\left( \sum_{j \in \mathcal{R}} \tau_j x^j \right).$$

Note that $\mathcal{I} \subseteq \mathcal{K} \cap \mathcal{R}$ and $\alpha_j = \tau_j$ when $j \in \mathcal{I}$.

We know how to estimate the sum over $\mathcal{R}$ by LLW, since $\tilde{\mathcal{R}} = \mathcal{R}$. 

Weyl-type inequality in $\mathbb{F}_q[t]$
Thank You!