A construction of Bowen-Margulis measure

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Anosov flows: Definition

Fix M to be a C^1 Riemannian manifold and $\{g_t\}$ a differentiable flow on M. We say g_t is an *Anosov flow* if there is a splitting of tangent bundle TM,

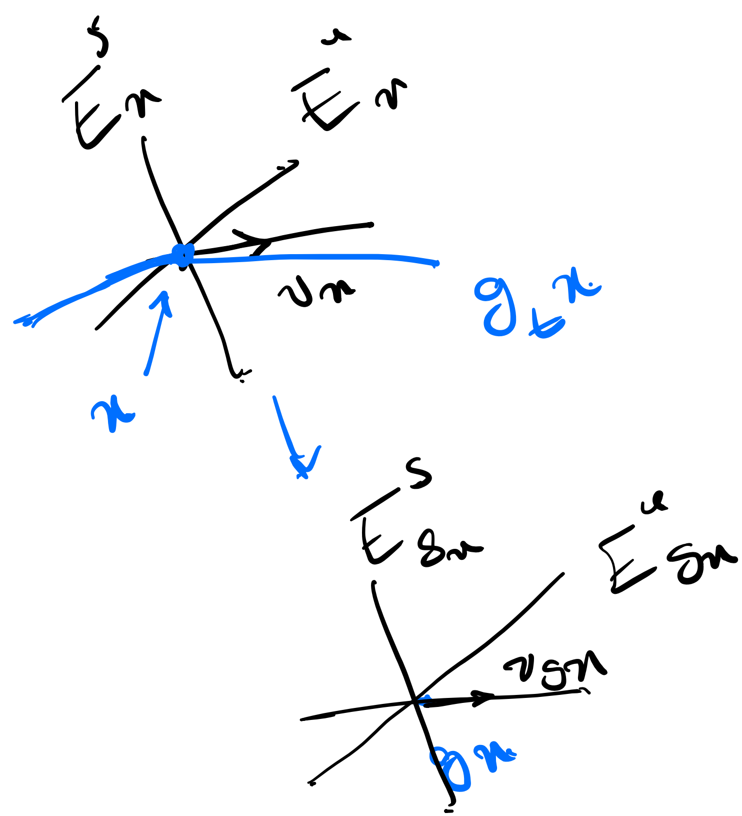
$$T_xM = E_x^{\mathrm{s}} \oplus E_x^{\mathrm{u}} \oplus \mathbb{R}v_x, \quad \text{for} \quad x \in M$$

such that

1. $v_x = \frac{dg_t x}{dt}|_{t=0}$ is the flow direction.
2. E^{s} and E^{u} are g_t–invariant.

and there is a constant $\kappa > 0$ such that

3. $\|D_x g_t(v)\| \leq e^{-\kappa t}\|v\|$ for $v \in E_x^{\mathrm{s}}$ and $t \geq 0$.
4. $\|D_x g_t(v)\| \geq e^{\kappa t}\|v\|$ for $v \in E_x^{\mathrm{u}}$ and $t \geq 0$.



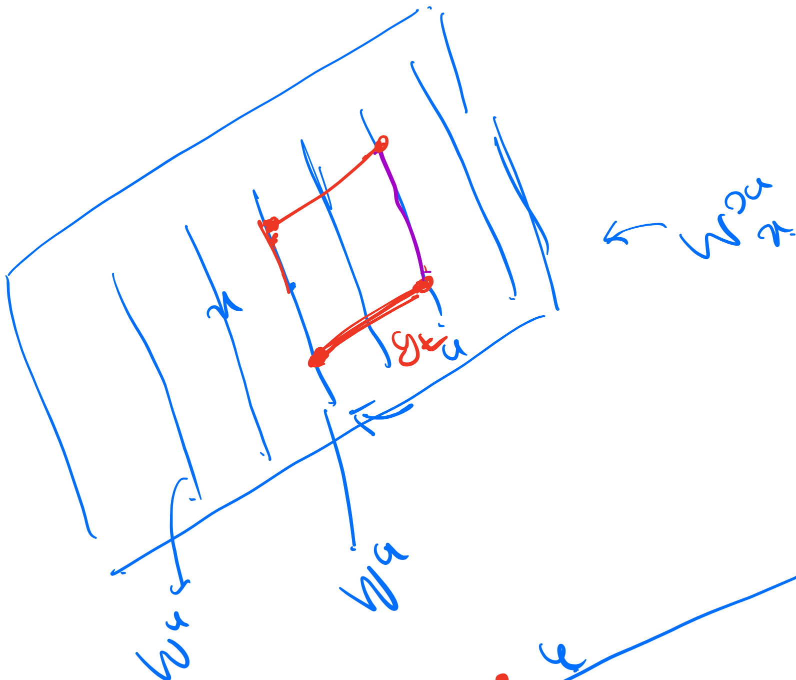
$$D_x g_t E_x^{s} = E_{g_t x}^{s}$$

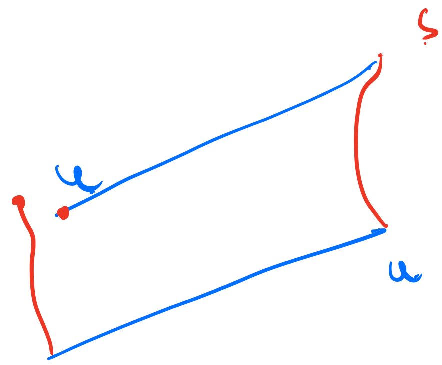
Anosov flows: Properties

strongly unstab

1. The distribution E^{u} is integrable, and the corresponding foliation is called the *unstable* foliation. The leaf that passes through $x \in M$ is denoted by W_x^{u}. W^u_x
2. The distribution $W^{\mathrm{u}} \oplus \mathbb{R}v$ is integrable, and the corresponding foliation is called the *center-unstable* foliation. The leaf that passes through $x \in M$ is denoted by $W_x^{0\mathrm{u}}$.
3. The distribution E^{s} is integrable, and the corresponding foliation is called the *stable* foliation. The leaf that passes through $x \in M$ is denoted by W_x^{s}
4. The distribution $E^{\mathrm{s}} \oplus \mathbb{R}v$ is integrable, and the corresponding foliation is called the *center-stable* foliation. The leaf that passes through $x \in M$ is denoted by W_x^{0s}.
5. The leaves of unstable foliation are C^1, and they vary in a Hölder-continuous way.

$T_y W^u_x = E^u_x$


x
W^{cu}_x
W^s
W^u
u


s
u
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$[\cdot,\cdot]$, $\theta(\cdot,\cdot)$, and flow box

Definition

The unstable metric d^{u} on W^{u} is defined by considering the restriction of Riemannian metric to unstable leaf W^{u}. This induces a topology on W^{u} that is different from the topology induced from M.

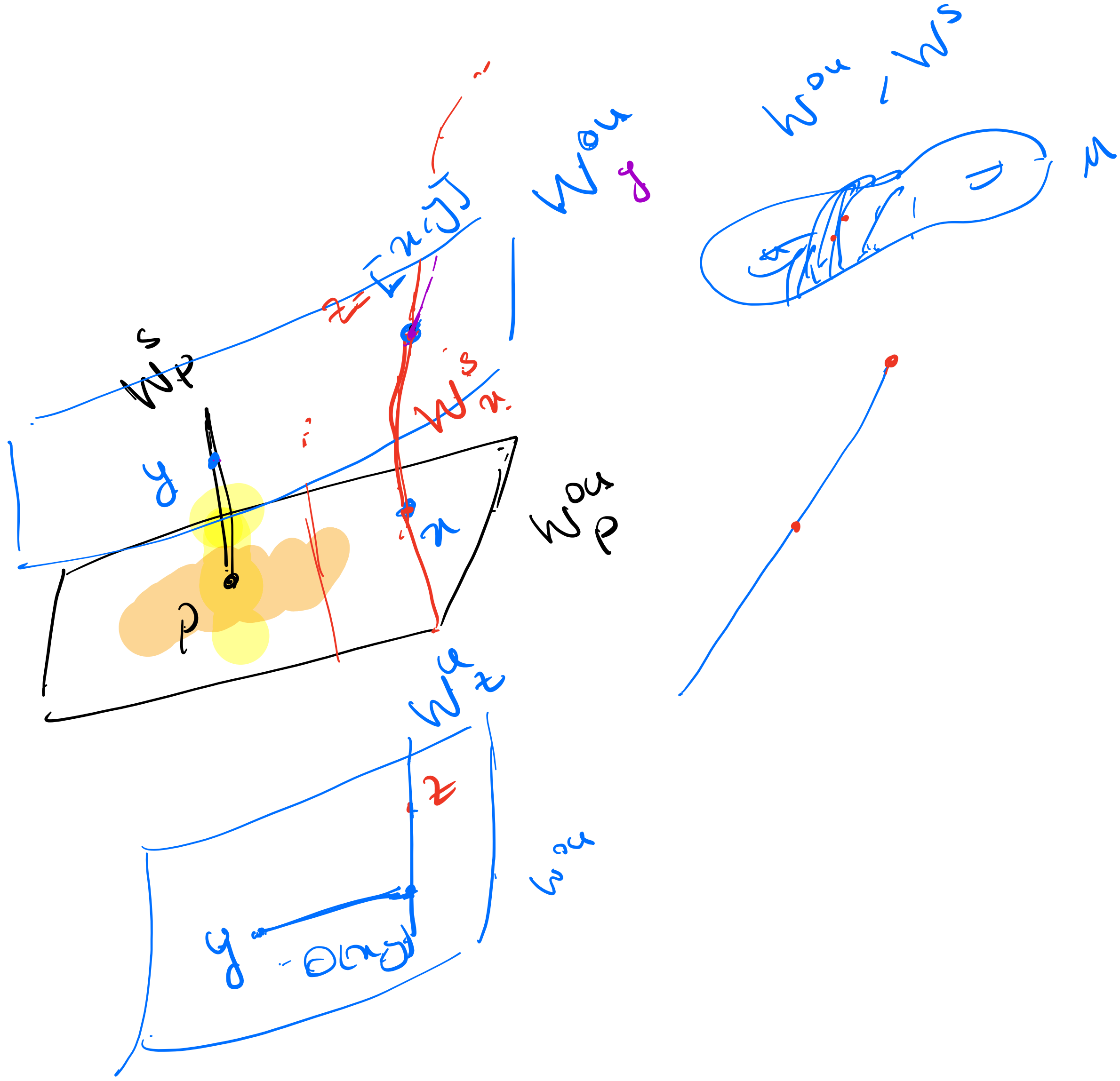
For $p \in M$ and $x \in W^{0\mathrm{u}}_x, y \in W^{\mathrm{s}}$ close to p,

1. Set $z := [x, y] := W_x^{\text{s, loc}} \cap W_y^{\text{0u, loc}}$.
2. Set $\theta(x, y) \in \mathbb{R}$ such that $g_t(y) \in W_z^{\mathrm{u}}$ for $t := \theta(x, y)$.

Definition

A set of the form $[U, V]$ for small neighborhoods $p \in U \subset W^{0\mathrm{u}}$ and $p \in V \subset W^{\mathrm{s}}$ is called a *flow box*.

"$\{[x,y] : x \in U, y \in V\}$"



W^s_p
y
p
W^{ou}_p
x
W^s_x
$z = [x, y]$
W^{ou}_y
W^u_z
z
y
$\Theta(x,y)$
W^{ou}
W^{ou}, W^s
M

Invariant measures

1. There exists a unique g_t–invariant measure in the Lebesgue class, called the *smooth invariant measure*.
2. Denoting the topological entropy of g_t by δ, there exists a unique g_t–invariant measure with entropy δ. This measure is called the Bowen-Margulis measure, and is denoted by μ_{BM} (or just μ).

(X, μ, T)

Gibbs measures

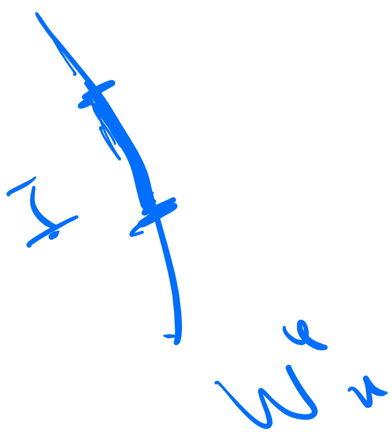
M, g_t, μ

$2.$

Two constructions of Bowen-Margulis measure

1. Bowen's construction.
2. Margulis' construction.

$G_T := \{\gamma \text{ closed orbit}, \ \ell(\gamma) \le T\}$

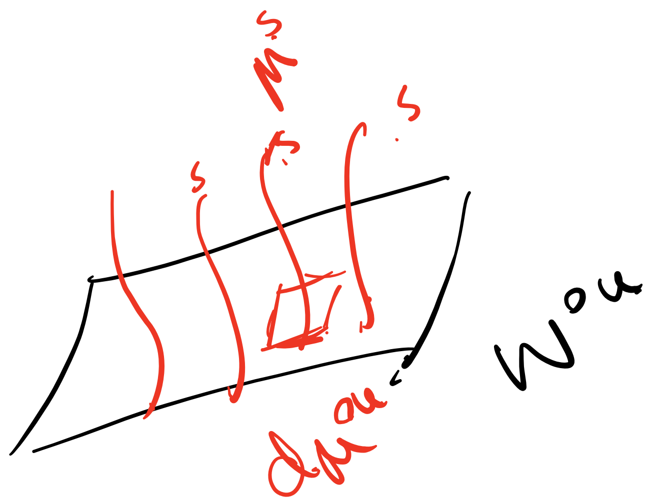


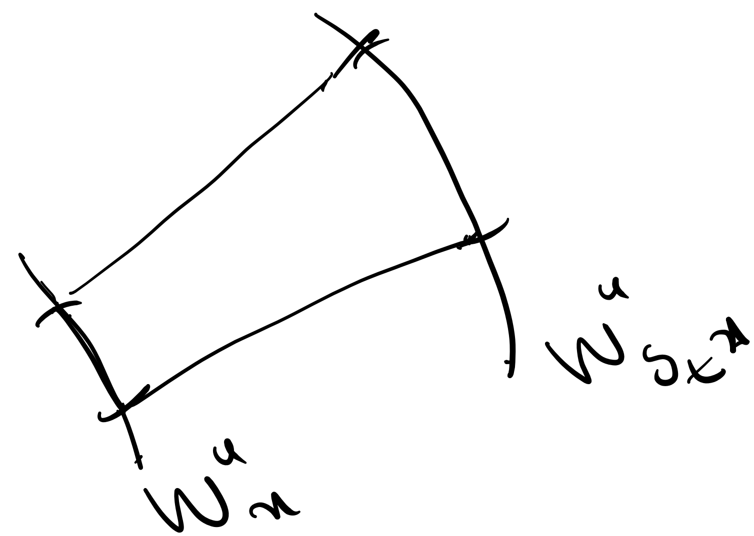
g_{-t}

Bowen-Margulis measure: Properties

There are a family of measures μ_x^{u} (resp. $\mu_x^{0\mathrm{u}}, \mu_x^{\mathrm{s}}$) supported on W_x^{u} (resp. $W_x^{0\mathrm{u}}, W_x^{\mathrm{s}}$) for $x \in M$ such that

1. $d\mu^{\mathrm{u}}(g_t x) = e^{\delta t}\mu^{\mathrm{u}}(x)$.
2. $d\mu^{0\mathrm{u}}(y) = e^{\delta t} d\mu^{\mathrm{u}}(x)\, dt$ for $y = (x, t)$.
3. Locally, $d\mu_{\mathrm{BM}} = d\mu^{0\mathrm{u}}\, d\mu^{\mathrm{s}}$.
4. In a flow box $[U, V]$, $\mu_{\mathrm{BM}}(z) = e^{-\delta\theta(x,y)} d\mu^{0\mathrm{u}}(x)\, d\mu^{\mathrm{s}}(y)$ for $z = [x, y]$.





The Presentation

We present the paper "A new description of the Bowen-Margulis measure" by Ursula Hamenstädt . The paper is written for g_t the geodesic flow on a manifold of (variable) negative curvature; however, the arguments carry over to general Anosov flows. We present the argument for Anosov flows.
To simplify the notation, we assume $\kappa = 1$ in the definition of Anosov flow. We also assume $\dim E^{\mathrm{s}} = \dim E^{\mathrm{u}} = 1$.

Notation. For functions $f(x), g(x)$ we write $f(x) \prec g(x)$ if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$ for all x in the domain. We write $f(x) \asymp g(x)$ if $f(x) \prec g(x)$ and $f(x) \succ g(x)$. We may extend this notation to measures.

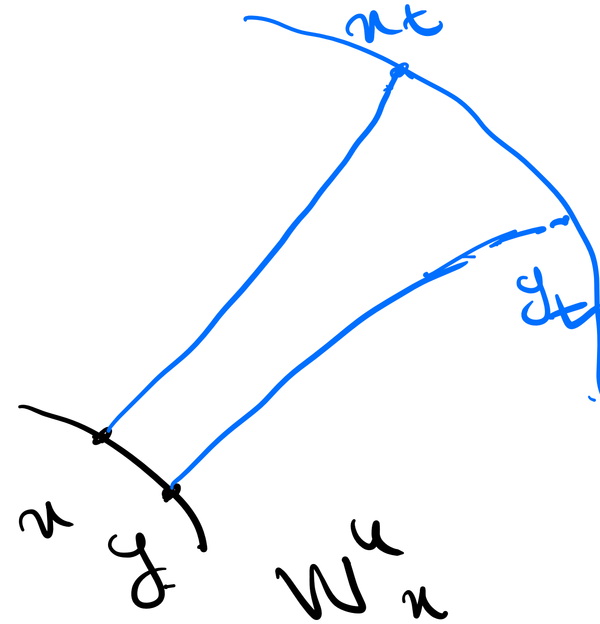
μ, ν on X, $\mu \prec \nu$ if $\exists\, C > 0$ $\forall$ for every $A \in \mathcal{B}(X)$, $\mu(A) \leq C \nu(A)$,

$\mu \precsim \nu$

Hamenstädt distance

Hamenstädt distance $d^{\mathrm{u}}_H(\cdot,\cdot)$ is defined on unstable foliation W^{u}.

1. The definition.
2. Verifying the triangle inequality.
3. For $r<1, x \in M$, $\mu^{\mathrm{u}}_{\mathrm{BM}}(B^{\mathrm{u}}_H(x,r)) \asymp r^{\delta}$. $= e^{-T\delta}$



$d^u_H(x,y)$

Fix $x,y \in W^u$,

$x_t := g_t x$, $y_t := g_t y$

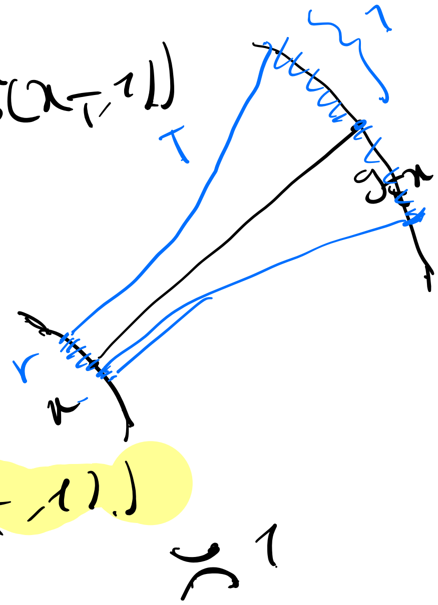
$d_t(x_t, y_t) := d^u(x_t, y_t)$

$\exists\, T := T(x,y)$ s.t. $d_T(x_T, y_T) = 1$,

we define $d^u_H(x,y) := e^{-T}$

Let T be such that $r = e^{-T}$

then $B^u_{Ham}(x, e^{-T}) = g_{-T}(B^u(x_T, 1))$



$\mu^u_{BM}(B^u_{Ham}(x, e^{-T})) =$

$e^{-\delta T} \mu_{BM}(B^u(x_T, 1))$ $\asymp 1$

$\{\mu(B^u(y, 1))\}_{y \in M}$

Hausdorff measure

For a metric space (X, d), the δ–dimensional Hausdorff measure μ_H is defined by

$$\mu_H(A) := \sup_{\epsilon \to 0} \inf\{\sum_{i=0}^{\infty} r_i^\delta \mid r_i \leq \epsilon, \text{ and } A \subset \bigcup B(x_i, r_i)\}$$

for A a Borel subset of X.

Definition

We denote the δ–dimensional Hausdorff measure on $\underline{W^{\mathrm{u}}}$ with respect to $\underline{d_H^{\mathrm{u}}}$ by $\underline{\mu_H^{\mathrm{u}}}$.

Theorem. $\mu^u_H = c\,\mu^u_{BM}$.

$\mu^{\mathrm{u}}_{\mathrm{BM}} \asymp \mu^{\mathrm{u}}_{H}$ $\exists C$

1. $\mu^{\mathrm{u}}_{\mathrm{BM}} \prec \mu^{\mathrm{u}}_{H}$.
2. $\mu^{\mathrm{u}}_{H} \prec \mu^{\mathrm{u}}_{\mathrm{BM}}$.

$$\frac{1}{C}\mu^{u}_{H}(A) \leq \mu^{u}_{Bm}(A) \leq C\mu^{u}_{H}(A)$$

$*\quad \mu^u_{BM} \overset{?}{\lesssim} \mu^u_H$

$\mu_{BM}(B^u_H(x,r)) \doteq r^\delta$

$A \subset \mathcal{B}(W^u_x)$, $A \subset \bigcup_i^\infty B^u_H(x_i, r_i)$

W^u

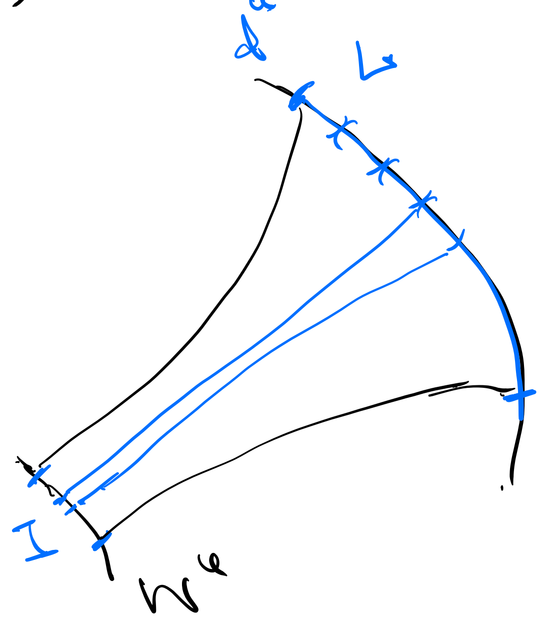
$\mu^u_{BM}(A) \leq \sum_i^\infty \mu_{BM}(B^u_H(x_i, r_i))$

$\lesssim \sum_{i=1}^\infty r_i^\delta$

$\mu^u_H \ll \mu^u_{BM}$

Construct "good" covers of $A \subset \mathcal{B}(W^u)$

Fix an interval $I \subset W^u$, let T be large enough.

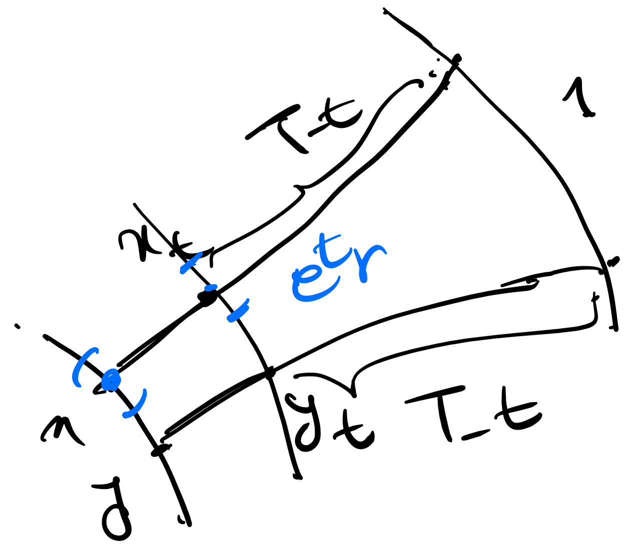


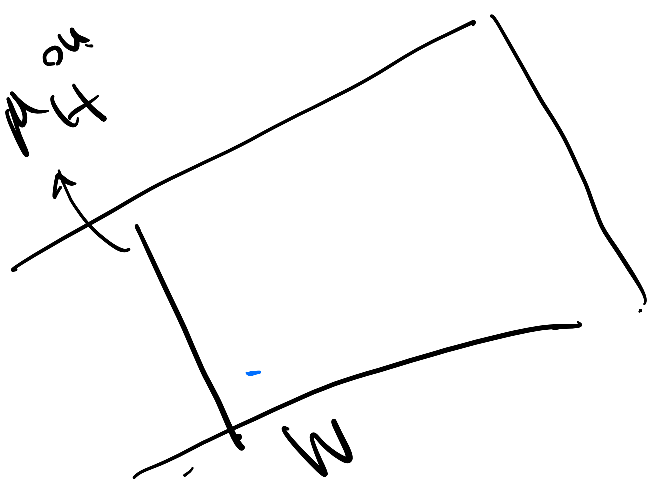
Hausdorff measure on M

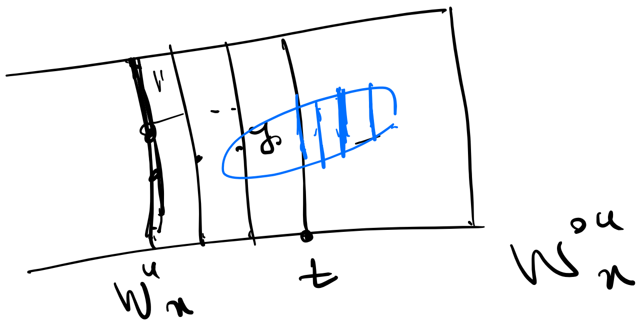
1. For $x, y \in W^{\mathrm{u}}$, $d_H^{\mathrm{u}}(g_t x, g_t y) = e^t d_H(x, y)$. Hence $d\mu_H^{\mathrm{u}}(g_t x) = e^{\delta t} d\mu_H^{\mathrm{u}}(x)$.
2. Define $\mu_H^{0\mathrm{u}}$ on $W^{0\mathrm{u}}$ by $d\mu_H^{0\mathrm{u}}(y) = e^{\delta t} d\mu_H^{\mathrm{u}}(x)\, dt$.
3. Define μ_H^{s}.
4. Define μ_H on M so that locally, $d\mu_H = d\mu_H^{0\mathrm{u}}\, d\mu_H^{\mathrm{s}}$. Then, in a flow box $[U, V]$, $\mu_H(z) = e^{-\delta\theta(x,y)} d\mu_H^{0\mathrm{u}}(x)\, d\mu_H^{\mathrm{s}}(y)$ for $z = [x, y]$.

$d_H(x,y) = e^{-T}$

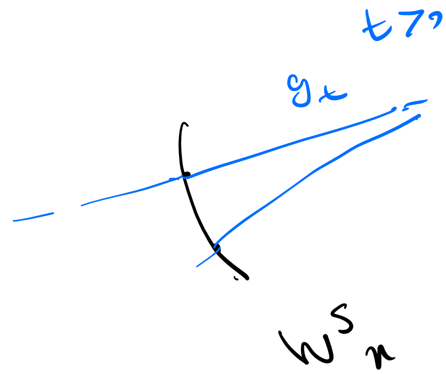
$d_H'(x_t, y_t) = e^{-(T-t)}$







$d\mu^u_H(y)$



$\mu^u_H \asymp \mu^u_{BM}$, $\mu^{0u}_H \asymp \mu^{0u}_{BM}$

$\mu^s_H \asymp \mu^s_{BM}$

$\mu_H \asymp \mu_{BM} \rightsquigarrow \dfrac{d\mu_H}{d\mu_{BM}}$ is

g_t-invariant $\to \mu_H = c\,\mu_{BM}$

μ_{BM} and μ_H are the same, up to a constant

1. There exists a constant $C > 0$ such that $\mu_H = C\mu_{\mathrm{BM}}$.
2. There are constants C^{u} and C^{s} such that $\mu_H^{\mathrm{u}} = C^{\mathrm{u}}\mu_{\mathrm{BM}}^{\mathrm{u}}$ and $\mu_H^{\mathrm{s}} = C^{\mathrm{s}}\mu_{\mathrm{BM}}^{\mathrm{s}}$.

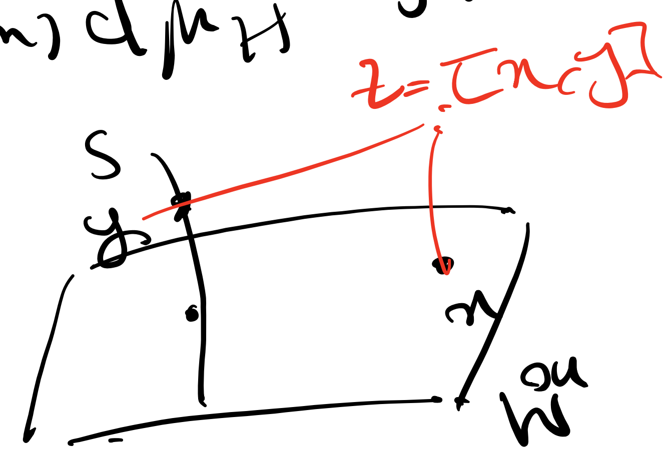
Question

Find $C^{\mathrm{u}}, C^{\mathrm{s}}$!

in a flow box we have

$$d\mu_{BM}(z) = e^{s\theta(x,y)}\, d\mu_{BM}^{ou}(x)\, d\mu_{BM}^{s}(y)$$

"$$d\mu_H(z) = e^{s\theta(x,y)}\, d\mu_H^{ou}(x)\, d\mu_H^{s}(y)$$"



$$\mu \otimes \nu \overset{c}{=} \mu' \otimes \nu' \to \mu = c_1 \mu', \quad \nu = c_2 \nu'$$