Oscillation results for the summatory functions of fake mu's

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Comparative Prime Number Theory Symposium
Jun 20, 2024

Fake μ's

Recently, Martin, Mossinghoff, and Trudgian (2023) investigated comparative number theoretic results for a family of arithmetic functions called "fake μ's":

Definition (Martin, Mossinghoff, and Trudgian (2023))

An arithmetic function f is a fake μ if:

- *f is a multiplicative function;*
- *For each positive integer j, $f(p^j) = \varepsilon_j \in \{-1, 0, 1\}$ holds for all primes p.*

We identify f with the defining sequence $(\varepsilon_j)_{j=1}^{\infty}$.

Question

What can say about the oscillation for the summatory function of a fake mu, that is, $\sum_{n \le x} f(n)$?

Mertens conjecture

- Let $\mu(n)$ be the Möbius function, and let $M(x) = \sum_{n \le x} \mu(n)$.
- In 1897, Mertens conjectured $|M(x)| \le \sqrt{x}$ for all $x \ge 1$; this was known as *Mertens' conjecture*.
- This conjecture was first disproved by Odlyzko and te Riele (1985).
- Hurst (2018):

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1.837625 \quad \text{and} \quad \limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1.826054.$$

- μ is a fake μ! Indeed, the corresponding sequence $(\varepsilon_j)_{j=1}^{\infty}$ satisfies $\varepsilon_1 = -1$ and $\varepsilon_j = 0$ for $j \ge 2$.

Pólya's problem

- Let $\lambda(n) = (-1)^{\Omega(n)}$ be the Liouville function, where $\Omega(n)$ is the number of prime factors of n counted with multiplicity.
- Let $L(x) = \sum_{n \leq x} \lambda(n)$.
- Pólya (1919) asked if $L(x) \leq 0$ holds for all x; this was known as the *Pólya problem*, often mistakenly named as *Pólya's conjecture*.
- The problem was first resolved in negative by Haselgrove (1958)
- Mossinghoff and Trudgian (2017):

$$\liminf_{x \to \infty} \frac{L(x)}{\sqrt{x}} < -2.3723 \quad \text{and} \quad \limsup_{x \to \infty} \frac{L(x)}{\sqrt{x}} > 1.0028$$

- λ is a fake μ: the corresponding sequence $(\varepsilon_j)_{j=1}^{\infty}$ satisfies $\varepsilon_j = (-1)^j$.

ABCPNT

- Mertens' conjecture and the Pólya problem motivated substantial work in comparative prime number theory.
- An annotated bibliography for comparative prime number theory (ABCPNT) [arXiv:2309.08729]
- Written by Greg Martin and a group of students in UBC.
- So far, 330 papers, 98 pages.
- Record every publication on the topic of comparative prime number theory together with a summary of its results, use a unified system of notation for the quantities being studied and for the hypotheses under which results are obtained.
- Send an email to Greg by June 30 if you have suggestions or comments!

Tanaka's Möbius function

- Tanaka's Möbius function: for integers $k \geq 2$, Tanaka (1980) defined the generalized Möbius function $\mu_k(n)$ to be $\mu_k(n) = (-1)^{\Omega(n)}$ if n is k-free and $\mu_k(n) = 0$ otherwise. Note that $\mu_2 = \mu$, and $\mu_\infty = \lambda$.
- Let $M_k(x) = \sum_{n \leq x} \mu_k(n)$. Tanaka showed that $M_k(x) - B_k\sqrt{x} = \Omega_\pm(\sqrt{x})$.

Theorem (Martin, Mossinghoff, and Trudgian (2023))

If f is a fake μ with $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$, then its summatory function $F(x)$ satisfies

$$F(x) - b\sqrt{x} = \Omega_\pm(\sqrt{x})$$

where b is twice the residue at $\frac{1}{2}$ of the Dirichlet series corresponding to $f(n)$.

They remarked that "a function with no bias at scale $\sqrt{x}$ could well see one at a smaller scale".

More fake μ's: k-free and k-full

- Indicator of k-free numbers: $\varepsilon_j = 1$ for $j < k$ and $\varepsilon_j = 0$ for $j \geq k$.
- Let $Q_k(x)$ be the number of k-free numbers up to x.
- $R_k(x) = Q_k(x) - x/\zeta(k)$.
- It is well-known that $R_k(x) = \Omega_{\pm}(x^{1/2k})$.
- Indicator of k-full numbers: $\varepsilon_j = 0$ for $j < k$ and $\varepsilon_j = 1$ for $j \geq k$.
- Let $N_k(x)$ be the number of k-full numbers up to x.
- It is known that $N_k(x)$ admits the asymptotic formula of the form

$$N_k(x) = \sum_{k \leq j \leq 4k+4} b_j x^{1/j} + \Delta_k(x)$$

- Bateman and Grosswald (1958)*: $\Delta_k(x) = \Omega(x^{1/(4k+4)})$.

Main result

Theorem (Martin, Y., 2024+)

Let f be a fake μ with the critical index ℓ. Then its summatory function

$$F(x) - \sum_{j=1}^{2\ell} \mathrm{Res}\left(T \cdot \frac{x^s}{s}, \frac{1}{j}\right) = \Omega_{\pm}(x^{\frac{1}{2\ell}}).$$

- Most residues on the above equation are probably simply 0
- The lower bound can be improved by a power of $\log x$ in certain cases.

*Exceptions:

- $\varepsilon_j \equiv 1$ and $\varepsilon_j \equiv 0$ (the identity function and the indicator function of $n = 1$, respectively).
- We also need to exclude the indicator function of k-th powers for $k \geq 2$, that is, $\varepsilon_j = 1$ if $k \mid j$ and $\varepsilon_j = 0$ otherwise.
- In these three cases, there is no oscillation result.

Critical index

- Given a fake μ function f defined via the sequence $(\varepsilon_j)_{j=1}^{\infty}$.
- Let

$$T(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

- If for $\sigma > 1$, we can write

$$T(s) = \frac{\prod_{j=1}^{\ell-1} \zeta(js)^{b_j}}{\zeta(\ell s)^{b_\ell}} V(s), \tag{1}$$

where $b_1, b_2, \ldots, b_{\ell-1}$ are non-negative integers, b_ℓ is a positive integer, and $V(s)$ is of the form

$$V(s) = \prod_p \left(1 + \sum_{j \geq 2\ell+1} \frac{\eta_j}{p^{js}}\right),$$

then the critical index of f is ℓ.

Outline of the proof: main term

For $\sigma > 1$, we can write

$$T(s) = U(s) \cdot \prod_{j=1}^{2\ell} \zeta(js)^{a_j}, \tag{2}$$

- $a_\ell < 0$ and $a_j = 0$ for $1 \leq j < \ell$.
- $U(s)$ analytic for $\sigma > \frac{1}{2\ell+1}$.
- Real poles with real parts at least $\frac{1}{2\ell}$ can only possibly occur at $s = 1, \frac{1}{2}, \cdots, \frac{1}{2\ell}$.
- They contribute to the main term.

Outline of the proof: error term

Proposition (Martin, Y., 2024+)

There is zero ρ of ζ such that $\Re(\rho) \geq \frac{1}{2}$ and

$$U\left(\frac{\rho}{\ell}\right) \cdot \prod_{\substack{1 \leq j \leq 2\ell \\ j \neq \ell}} \zeta\left(\frac{j\rho}{\ell}\right) \neq 0. \tag{3}$$

- This guarantees that ρ/ℓ is indeed a pole of $T(s)$, which contributions to the oscillation of the error term.
- Apply Landau's theorem to get the Omega result.
- Proof: zero density estimate+ reduction to truncated Euler products+ Landau's formula.

Example

- Consider the case $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$.
- $$T(s) = \frac{U(s)\zeta(2s)}{\zeta(s)}, \quad U(s) = \prod_p \left(1 + \sum_{j \geq 3} \frac{\varepsilon_{j-1} + \varepsilon_j}{p^{js}}\right)$$
- The critical index is $\ell = 1$.
- Suffices to show $U(\rho_1)\zeta(2\rho_1) \neq 0$, where $\rho_1 \approx \frac{1}{2} + 14.134725i$.
- Suffices to show $U(\rho_1) \neq 0$
- This can be done using the triangle inequality via case-by-case analysis.

Algorithm to compute the critical index

$c_1 \leftarrow$ the smallest i such that $\varepsilon_i \neq 0$
if $\varepsilon_{c_1} = -1$ **then**
  $M \leftarrow 0$
  $\ell \leftarrow c_1$
  return ℓ
$m \leftarrow 1$
while *true* **do**
  $j \leftarrow c_m + 1$
  while *true* **do**
    $n_j \leftarrow$ the number of representations of j from$\{c_1, c_2, \ldots, c_m\}$
    if $n_j = 0$ *and* $\varepsilon_j = 1$ **then**
      $c_{m+1} \leftarrow j$
      break
    if $n_j > \varepsilon_j$ **then**
      $M \leftarrow m$
      $\ell \leftarrow j$
      return ℓ
    $j \leftarrow j + 1$
  $m \leftarrow m + 1$

Factorization

Theorem (Martin, Y., 2024+)

We have

$$T(s) = U(s) \cdot \frac{\prod_{j=1}^{M} \zeta(c_j s)}{\zeta(\ell s)^{n_\ell - \varepsilon_\ell}} \cdot \prod_{j=\ell+1}^{2\ell} \zeta(js)^{a_j}, \tag{4}$$

where

$$a_j = \begin{cases} \sum_{I \subset [M]} (-1)^{\# I} \varepsilon_{j - \sum_{i \in I} c_i}, & \ell + 1 \leq j \leq 2\ell - 1 \\ -\frac{(\varepsilon_\ell - n_\ell)^2 + \varepsilon_\ell - n_\ell}{2} + \sum_{I \subset [M]} (-1)^{\# I} \varepsilon_{2\ell - \sum_{i \in I} c_i} & j = 2\ell \end{cases}, \tag{5}$$

and $U(s)$ *is analytic for* $\sigma > \frac{1}{2\ell+1}$.

$$F(x) - \sum_{j=1}^{M} \mathrm{Res}\left(T \cdot \frac{x^s}{s}, \frac{1}{c_j}\right) - \sum_{j=\ell+1}^{2\ell} \mathrm{Res}\left(T \cdot \frac{x^s}{s}, \frac{1}{j}\right) = \Omega_{\pm}\left(x^{\frac{1}{2\ell}} (\log x)^{n_\ell - \varepsilon_\ell - 1}\right).$$

Thank you for your attention!