Local statistics for zeros of Artin-Schreier L-functions

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Joint work with Noam Pirani (TAU)

Artin-Schreier curves

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An *Artin-Schreier curve* is a curve defined by an affine equation

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over a field F of characteristic p, where $f \in F(x)$ is a rational function not of the form $f = h^p - h$, $h \in \overline{F}(x)$.

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Denote by C_f the smooth projective model of the curve defined by $y^p - y = f(x)$.

Artin-Schreier L-functions

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Now consider $f \in \mathbb{F}_q(x)$, $f \neq h^p - h\,(h \in \overline{\mathbb{F}}_q(x))$. The zeta-function of C_f factors as follows:

$$\zeta(u, C_f) := \exp\left(\sum_{r=1}^{\infty} \frac{\# C_f(\mathbb{F}_{q^r})}{r} u^r\right)$$

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$$L(u, f, \psi) = \exp\left[\sum_{r=1}^{\infty} \sum_{\substack{\alpha \in \mathbb{F}_{q^r} \cup \{\infty\} \\ f(\alpha) \neq \infty}} \psi\left(\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p} f(\alpha)\right) \frac{u^r}{r}\right]$$

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is called the Artin-Schreier L-function associated with f, ψ.

Artin-Schreier L-functions: basic properties

- $L(u, f, \psi)$ is a polynomial of degree $\frac{2\mathfrak{g}(C_f)}{p-1}$ ($\mathfrak{g}$ denotes the genus).

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- Riemann Hypothesis (proved by Weil):

$$L(u, f, \psi) = \prod_{j=1}^{2\mathfrak{g}(C_f)/(p-1)} \left(1 - q^{1/2} e(\theta_j(f))u\right), \quad \theta_j(f) \in \mathbb{R},\ 1 \leq j \leq \frac{2\mathfrak{g}(C_f)}{p-1}$$

$[e(t) = exp(2\pi t)]$.

Three families of Artin-Schreier L-functions

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1. The *polynomial* A-S family (assume $(d, p) = 1$):

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2. The *odd polynomial* A-S family (assume $(d, 2p) = 1$):

$$\mathcal{AS}_d^{0,\mathrm{odd}} = \{f \in \mathbb{F}_q[x] : \deg f = d,\ f(x) = -f(-x)\} \subset \mathcal{AS}_d^0.$$

Three families of Artin-Schreier L-functions (cont'd)

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3. The *ordinary* A-S family:

$$\mathcal{AS}_d^{\mathrm{ord}} = \left\{ f = \frac{h}{g} : h, g \in \mathbb{F}_q[x], (g,h) = 1, g \text{ squarefree}, \right.$$
$$\left. \deg f := \max(\deg h, \deg g) = d, \deg g \in \{d, d-1\} \right\}.$$

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$$\mathfrak{g}(C_f) = (p-1)(d-1), \quad \deg L(u, f, \psi) = 2(d-1).$$

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These are models in the following sense: if we choose $f \in \mathcal{F}_d$ uniformly in one of the above families and write $L(u, f, \psi) = \prod_{j=1}^{N} \left(1 - q^{1/2} e(\theta_j(f))u\right)$ and similarly for a random $A \in \mathcal{G}$ ($\mathcal{G}$ the corresponding compact classical group) let $e(\theta_j(A))$ be its eigenvalues. Then the collections $\theta_1(f), \dots, \theta_N(f) \in \mathbb{R}/\mathbb{Z}$ should behave statistically like the collections $\theta_1(A), \dots, \theta_N(A)$.

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Theorem (Katz)

Fix d. Let $\mathcal{F}_d$ be one of the above families and $\mathcal{G}$ the corresponding matrix group. Let $\phi : (\mathbb{R}/\mathbb{Z})^N \to \mathbb{C}$ ($N = \deg L(u, f, \psi)$ for any $f \in \mathcal{F}_d$) be a symmetric continuous function. Then

$$\lim_{q \to \infty} \langle \phi(\theta_1(f), \ldots \theta_N(f)) \rangle_{\mathcal{F}_d} = \langle \phi(\theta_1(A), \ldots, \theta_N(A)) \rangle_{\mathcal{G}}$$

(average w.r.t. Haar measure).

We will be interested in the q fixed, $d \to \infty$ regime!

Local statistics: n-level density

n-level density captures the local statistics of low-lying zeros at the scale of the average spacing between zeros, i.e. $(p-1)/2\mathfrak{g} = \deg L(u, f, \psi)^{-1}$ $(\mathfrak{g} = \mathfrak{g}(C_f))$.

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Let $\phi(\mathbf{t}) \in \mathcal{S}(\mathbb{R}^n)$ be a fixed (Schwartz) test function,

$$\phi_{2\mathfrak{g}/(p-1)}(\mathbf{t}) = \sum_{\mathbf{i}\in\mathbb{Z}^n} \phi\left(\frac{2\mathfrak{g}}{p-1}(\mathbf{t}+\mathbf{i})\right) \in \mathcal{C}^{\infty}((\mathbb{R}/\mathbb{Z})^n)$$

the associated periodic test function at scale $(p-1)/2\mathfrak{g}$.

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The n-level density is defined by

$$W_n(f;\phi) = \sum_{\substack{1\le i_1,\dots,i_n\le \frac{2\mathfrak{g}}{p-1} \\ \text{distinct}}} \phi_{2\mathfrak{g}/(p-1)}\left(\theta_{i_1}(f),\dots,\theta_{i_n}(f)\right).$$

Local statistics: n-level density, the random matrix case

For a unitary matrix A define $W_n(A;\phi)$ similarly.

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Theorem (Katz-Sarnak)

1. $$\lim_{N\to\infty}\langle W_n(A;\phi)\rangle_{\mathrm{U}(N)} = \int_{\mathbb{R}^n}\phi(\mathbf{t})R_n^{\mathrm{U}}(\mathbf{t})\mathrm{d}\mathbf{t},$$

   where
   $$R_n^{\mathrm{U}}(t_1,\dots,t_n) = \det\left(\frac{\sin\pi(t_i-t_j)}{\pi(t_i-t_j)}\right)_{1\le i,j\le n}.$$

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Local statistics: n-level density, main conjectures for A-S L-functions

Conjecture

1. $$\lim_{d\to\infty}\langle W_n(f;\phi)\rangle_{f\in\mathcal{AS}_d^0}=\int_{\mathbb{R}^n}\phi(\mathbf{t})R_n^{\mathrm{U}}(\mathbf{t})\mathrm{d}\mathbf{t}.$$

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2. $$\lim_{d\to\infty} \langle W_n(f;\phi)\rangle_{f\in\mathcal{AS}_d^{0,\mathrm{odd}}} = \int_{\mathbb{R}^n} \phi(\mathbf{t}) R_n^{\mathrm{USp}}(\mathbf{t})\mathrm{d}\mathbf{t}.$$

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Theorem (E. '12)

Assume $\operatorname{supp}\hat{\phi} \subset (-2(1-1/p), 2(1-1/p))$. *Then*

$$\lim_{d\to\infty} \langle W_1(f;\phi)\rangle_{\mathcal{AS}_d^0} = \int_{-\infty}^{\infty} \phi(t)\mathrm{d}t = \langle W_1(A;\phi)\rangle_{U_{d-1}}.$$

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Assume $\operatorname{supp}\hat{\phi} \subset \{|\tau| + |\sigma| < 1 - 1/p\}$. *Then*

$$\lim_{d\to\infty} \langle W_2(f;\phi)\rangle_{\mathcal{AS}_d^0} = \iint_{\mathbb{R}^2} \phi(t,s)\left(1 - \left(\frac{\sin\pi(t-s)}{\pi(t-s)}\right)^2\right)\mathrm{d}t\mathrm{d}s =$$
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Bucur, David, Feigon, Lalín and Sinha studied *mesoscopic* statistics of zeros (i.e. at the scale $\omega(d)\frac{p-1}{2\mathfrak{g}}$ where $\omega(d) \to \infty$) for several families of A-S L-functions including $\mathcal{AS}_d^{\mathrm{ord}}$, obtaining central limit theorems for the number of zeros in mesoscopic intervals.

New results

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Theorem (E., Pirani '21; improved result for 2-level density of polynomial A-S family)

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Theorem (E., Pirani '21; first zero-density result for odd polynomial A-S family)

Assume $\operatorname{supp}\hat{\phi} \subset (-(1-1/p),(1-1/p))$. *Then*

$$\lim_{d\to\infty} \langle W_1(f;\phi)\rangle_{\mathcal{AS}_d^{0,\mathrm{odd}}} = \int_{-\infty}^{\infty} \phi(t)\left(1 - \frac{\sin 2\pi t}{2\pi t}\right) \mathrm{d}t =$$

$$= \lim_{d\to\infty} \langle W_1(A;\phi)\rangle_{\mathrm{USp}(d-1)}.$$

New results (cont'd)

Theorem (E., Pirani '21; first zero-density result for ordinary A-S family)

Assume $\operatorname{supp}\hat{\phi} \subset (-1,1)$. *Then*

$$\lim_{d\to\infty} \langle W_1(f;\phi)\rangle_{\mathcal{AS}_d^{\mathrm{ord}}} = \int_{-\infty}^{\infty} \phi(t)\mathrm{d}t = \langle W_1(A;\phi)\rangle_{\mathrm{U}(2d-2)}.$$

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$$\{L(u,f,\psi) : f \in \mathcal{H}_g\} = \{(1-u)^{-1}L(u,\chi) : \chi \text{ primitive char. mod } g^2,\ \chi^p = 1\}.$$

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Step II: apply Fourier series to $\phi_{2\mathfrak{g}/(p-1)}$ + the condition $\operatorname{supp}\hat{\phi} \subset (-1,1)$ + explicit formula + orthogonality relation for characters, to express $\langle W_1(f;\phi)\rangle_{\mathcal{H}_g}$ as a sum over primes.

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$$W_1(f;\phi) = \frac{1}{2d-2} \sum_{-(1-\delta)(2d-2)\le r \le (1-\delta)(2d-2)} \hat{\phi}\left(\frac{r}{2d-2}\right) \sum_{j=1}^{2d-2} e(r\theta_j(f)) =$$

$$= \frac{1}{2d-2} \sum_{r=0}^{(1-\delta)(2d-2)} \Bigg[-2q^{-r/2} -$$

$$-q^{-r/2} \sum_{\substack{\deg c = r \\ \text{monic}}} \left(\hat{\phi}\left(\frac{r}{2d-2}\right) \chi_f(c) + \hat{\phi}\left(\frac{-r}{2d-2}\right) \overline{\chi}_f(c) \right) \Lambda(c) \Bigg]$$

(here $\delta > 0$ is such that $\operatorname{supp}\hat{\phi} \subset [-1+\delta, 1-\delta]$).

Take the average $\langle\cdot\rangle_{\mathcal{H}_g}$, which is the same as averaging over all primitive χ mod $g^2, \chi^p = 1$ and use the orthogonality relation:

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$$= \frac{1}{2d-2} \sum_{r=0}^{(1-\delta)(2d-2)} \Bigg[-2q^{-r/2} -$$

$$-q^{-r/2} \sum_{\substack{\deg c=r \\ \text{monic}}} \left(\hat{\phi}\left(\frac{r}{2d-2}\right) \chi_f(c) + \hat{\phi}\left(\frac{-r}{2d-2}\right) \overline{\chi}_f(c) \right) \Lambda(c) \Bigg]$$

(here $\delta > 0$ is such that $\operatorname{supp}\hat{\phi} \subset [-1+\delta, 1-\delta]$).

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Take the average $\langle\cdot\rangle_{\mathcal{H}_g}$, which is the same as averaging over all primitive $\chi \bmod g^2, \chi^p = 1$ and use the orthogonality relation:

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$\hat{\phi}(0) = \int_{\mathbb{R}} \phi(t)\mathrm{d}t$ is the desired main term. Ignoring the contribution of imprimitive characters (which can be dealt with similarly) it remains to show that

$$q^{-r/2} rd \# \left\{ c \in \mathbb{F}_q[x] \text{ prime}, \deg c = r : c \equiv u^p \pmod{g^2} \text{ for some } u \right\} = o(1)$$

whenever (crucially!) $r \leq (1-\delta)(2d-2)$ (recall $d = \deg g$).

Methods (cont'd)

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Another key observation:

$$\Lambda_g = \{c \in \mathbb{F}_q[x] : g | c'\} \subset \mathbb{F}_q[x]$$

is a free $\mathbb{F}_q[x^p]$-submodule of $\mathbb{F}_q[x]$ of rank $p-1$, i.e. an $\mathbb{F}_q[x^p]$-lattice of rank $p-1$.

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Remark: our results for the polynomial and odd polynomial families also crucially use the theory of reduced bases for $\mathbb{F}_q[x]$-lattices!

Suggested open problems: zero density and moments

[green = easy, blue = challenging, red = hard, black = 💀]

$M_k = \langle L(q^{-1/2}, f, \psi)^k \rangle$ (or $\langle |L(q^{-1/2}, f, \psi)|^k \rangle$ if k is even).

$\mathcal{AS}^{\mathrm{ord}}$: compute 1-level density for $\operatorname{supp} \hat{\phi} \subset (-1-\delta, 1+\delta), \delta > 0$ or $(-2-\delta, 2+\delta), \delta > 0$.
Compute $M_1, M_2, M_3, M_4, M_5, M_6$.

$\mathcal{AS}^{0}$: compute 1-level density for $\operatorname{supp} \hat{\phi} \subset (-(2-2/p)-\delta, 2-2/p+\delta)$.
Compute M_1, M_2, M_3, M_4.

$\mathcal{AS}^{0,\mathrm{odd}}$: compute 1-level density for $\operatorname{supp} \hat{\phi} \subset (-(1-1/p)-\delta, 1-1/p+\delta)$.
Compute M_1, M_2.

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A. Bucur, C. David, B. Feigon, M. Lalín, K. Sinha, *Distribution of zeta zeroes of Artin-Schreier covers*, Math. Res. Lett. 19 (2012), no. 6, pp. 1329-1356.

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Thank you!