

Upper bounds for negative moments of the Riemann zeta function

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Introduction

For $\Re s > 1$,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1}.$$

- It has a meromorphic continuation to \mathbb{C} with a simple pole at $s = 1$.
- It satisfies a functional equation $\zeta(s) \leftrightarrow \zeta(1 - s)$
- Trivial zeros at $s = -2m, m \geq 1$.
- **(RH)** The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.
- **Lindelöf hypothesis** $|\zeta(1/2 + it)| \ll t^\epsilon, \forall \epsilon > 0$.

Moments of $\zeta(s)$

- (RH) The non-trivial zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$.
- (Lindelöf hypothesis) $|\zeta(1/2 + it)| = O(t^\epsilon), \forall \epsilon > 0$.
- Hardy and Littlewood (1916): moments of $\zeta(s)$

$$I_k(T) = \int_0^T |\zeta(1/2 + it)|^{2k} dt$$

Lindelöf hypothesis $\iff I_k(T) \ll T^{1+\epsilon}, k = 1, 2, \dots$

Moments of $\zeta(s)$

Conjecture (Keating, Snaith)

For $k > 0$,

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt \sim a_k g_k T (\log T)^{k^2},$$

where

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m! \Gamma(k)}\right)^2 p^{-m},$$

$$g_k = ???.$$

- Heuristic idea:

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt \sim \int_0^T \sum_{m,n} \frac{d_k(m) d_k(n)}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it} dt.$$

- Keep $m = n$.

$$\sum_{n \leq T} \frac{d_k(n)^2}{n} \sim a_k (\log T)^{k^2}.$$

Moments of $\zeta(s)$

$$g_k = ???.$$

- $g_1 = 1$ (Hardy, Littlewood)
- $g_2 = \frac{1}{12}$ (Ingham)
- $g_3 = \frac{42}{9!}$ (conjecture Conrey, Gosh)
- $g_4 = \frac{24024}{16!}$ (conjecture Conrey, Gonek)

Conjecture (Keating, Snaith)

$$RMT: g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}.$$

Hybrid Model

- Assume RH. Let $\rho_n = 1/2 + \gamma_n$ denote the non-trivial zeros of $\zeta(s)$.

$$\zeta(1/2 + it) \approx \prod_{p \leq x} (1 - p^{-1/2-it})^{-1} \prod_{\substack{\gamma_n \\ |\gamma_n - t| < \frac{1}{\log x}}} \left(1 - \frac{1/2 + it}{\rho_n}\right).$$

Assume

$$\begin{aligned} \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} dt &\sim \left(\frac{1}{T} \int_0^T \left| \prod_{p \leq x} (1 - p^{-1/2-it}) \right|^{-2k} dt \right) \\ &\times \left(\frac{1}{T} \int_0^T \left| \prod_{\substack{\gamma_n \\ |\gamma_n - t| < \frac{1}{\log x}}} \left(1 - \frac{1/2 + it}{\rho_n}\right) \right|^{-2k} dt \right) \end{aligned}$$

Conjecture (Gonek, Hughes, Keating)

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt \sim a_k g_k T (\log T)^{k^2}.$$

Conjecture (Conrey, Farmer, Keating, Rubinstein, Snaith)

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt = TP_k(\log T) + O(T^{1-\delta}),$$

where P_k is a polynomial of degree k^2 and $\delta > 0$.

The same conjecture can be obtained by work of Conrey-Keating using long Dirichlet polynomials and results on divisor correlations.

- $k = 1$: Hardy, Littlewood (1916)
- $k = 2$: Ingham (1932), Heath-Brown (1979)
- $k = 3$: Ng (2016), conditional on conjectures about ternary additive divisor sums
- $k = 4$: Ng, Shen, Wong (2022), conditional on conjectures about quaternary additive divisor sums

Moments of $\zeta(s)$

- Under RH, lower bounds of the right order of magnitude for all $k > 0$, due to Ramachandra and Heath-Brown
- Unconditional sharp lower bounds for $k \geq 1$, due to Radziwill-Soundararajan
- Unconditional sharp lower bounds for $0 < k < 1$, due to Heap-Soundararajan
- Under RH, upper bounds of the right order of magnitude for $0 \leq k \leq 2$ due to Ramachandra and Heath-Brown
- Sharp upper bounds for $k = 1/n$ due to Heath-Brown, and for $k = 1 + 1/n$, due to Bettin-Chandee-Radziwill
- Unconditional sharp upper bounds for $0 \leq k \leq 2$, due to Heap-Radziwill-Soundararajan

Moments of $\zeta(s)$

Theorem (Soundararajan)

Assume the RH. Then for all positive real k and any $\epsilon > 0$,

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt \ll T(\log T)^{k^2 + \epsilon}.$$

Theorem (Harper)

Assume the RH. Then for all positive real k ,

$$\int_0^T |\zeta(1/2 + it)|^{2k} dt \ll T(\log T)^{k^2}.$$

Negative moments of $\zeta(s)$

Conjecture (Gonek, 1989)

Let $k > 0$ be fixed. Uniformly for $\frac{1}{\log T} \leq \delta \leq 1$, we have

$$I_{-k}(\delta, T) = \frac{1}{T} \int_1^T \left| \zeta\left(\frac{1}{2} + \delta + it\right) \right|^{-2k} dt \asymp \left(\frac{1}{\delta}\right)^{k^2},$$

and uniformly for $0 < \delta \leq \frac{1}{\log T}$, we have

$$I_{-k}(\delta, T) dt \asymp \begin{cases} (\log T)^{k^2} & \text{if } k < 1/2 \\ \log(e/(\delta \log T)) (\log T)^{k^2} & \text{if } k = 1/2 \\ (\delta \log T)^{1-2k} (\log T)^{k^2} & \text{if } k > 1/2. \end{cases}$$

- Random matrix theory computations (Berry-Keating; Forrester-Keating) suggest transition regimes when $k = (2n + 1)/2$, for n a positive integer

"Conjectures"

Conjecture

For $\frac{1}{\log T} \leq \delta \leq 1$, we have

$$I_{-k}(\delta, T) \sim a_k \left(\frac{1}{\delta}\right)^{k^2},$$

and for $0 < \delta \leq \frac{1}{\log T}$, we have

$$I_{-k}(\delta, T) \sim$$

$$\begin{cases} a_k (\log T)^{k^2} (\delta \log T)^{-j(2k-j)} & \text{if } j - \frac{1}{2} < k < j + \frac{1}{2} \\ a_k \log\left(\frac{e}{\delta \log T}\right) (\log T)^{k^2} (\delta \log T)^{-j(j-1)} & \text{if } k = j - \frac{1}{2} \ (j \geq 1), \end{cases}$$

and

$$a_k = \prod_p \left(1 - \frac{1}{p^{1+2\delta}}\right)^{k^2} \left(1 + \sum_{j=1}^{\infty} \frac{\mu_k(p^j)^2}{p^{(1+2\delta)j}}\right).$$

Negative moments of $\zeta(s)$

- Gonek obtained **lower bounds** consistent with the conjecture for all $k > 0$ in the range $\frac{1}{\log T} \leq \delta \leq 1$ and for $k < 1/2$ in the range $0 < \delta \leq \frac{1}{\log T}$.

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$$J_{-k}(T) = \sum_{0 < \Im \rho \leq T} |\zeta'(\rho)|^{-2k}.$$

- Under RH and the assumption that all zeros are simple, Gonek showed that

$$J_{-1}(T) \gg T.$$

- **Upper bounds?**

Upper bounds for negative moments

Theorem (Bui-F., 2022)

Assume RH. Let $k > 0$ and $\delta \gg \frac{1}{(\log T)^{\frac{1}{2k}-\epsilon}}$. Then

$$I_{-k}(\delta, T) \ll (\log T)^{O(1)}.$$

If $\delta = o\left(\frac{1}{(\log T)^{\frac{1}{2k}-\epsilon}}\right)$, and $\log(1/\delta) \ll \log \log T$, then

$$I_{-k}(\delta, T) \ll T^{\frac{k \log(1/\delta)}{\log \log T} - \frac{1}{2} + \epsilon}.$$

Negative moments

Corollary (Bui-F., 2022)

Assume RH. Then for $k > 0$

- $k \geq 1/2$ and $\delta \gg \frac{1}{(\log T)^{\frac{1}{2k}-\epsilon}}$,
- $k < 1/2$ and $\delta \log T \rightarrow \infty$

we have

$$I_{-k}(\delta, T) = \zeta(1 + 2\delta)^{k^2} a_k(1 + o(1)),$$

where $a_k = \prod_p \left(1 - \frac{1}{p^{1+2\delta}}\right)^{k^2} \left(1 + \sum_{j=1}^{\infty} \frac{\mu_k(p^j)^2}{p^{(1+2\delta)j}}\right)$.

Remark

The error term above can be written more explicitly as $T^{1-2\delta}(\log T)^{O(1)}$.

Remark

Similar results were obtained for negative moments in the family of quadratic L -functions over function fields.

The Ratios Conjecture for $\zeta(s)$

Conjecture (Farmer, 1993)

For $s = 1/2 + it$ and complex numbers $\alpha, \beta, \gamma, \delta$ of size $c/\log T$, such that $\Re\alpha, \Re\beta, \Re\gamma, \Re\delta > 0$ we have

$$\frac{1}{T} \int_0^T \frac{\zeta(s + \alpha)\zeta(1 - s + \beta)}{\zeta(s + \gamma)\zeta(1 - s + \delta)} dt \sim 1 + (1 - T^{-\alpha-\beta}) \frac{(\alpha - \gamma)(\beta - \delta)}{(\alpha + \beta)(\gamma + \delta)}.$$

- The conjecture implies many interesting results about zeros of $\zeta(s)$, such as the pair correlation conjecture of Montgomery.
- By adapting the “recipe” used by Conrey, Farmer, Keating, Rubinstein and Snaith to conjecture asymptotic formulas for moments of L -functions, one can make the following conjecture.

Conjecture (Conrey, Farmer, Zirnbauer, 2007)

$$\begin{aligned} & \int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)} dt \\ &= \int_0^T \left(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A(\alpha, \beta, \gamma, \delta) \right. \\ & \left. + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)} A(-\beta, -\alpha, \gamma, \delta) \right) dt + O\left(T^{1/2+\epsilon}\right), \end{aligned}$$

where

$$A(\alpha, \beta, \gamma, \delta) = \prod_p \frac{\left(1 - \frac{1}{p^{1+\gamma+\delta}}\right) \left(1 - \frac{1}{p^{1+\beta+\gamma}} - \frac{1}{p^{1+\alpha+\delta}} + \frac{1}{p^{1+\gamma+\delta}}\right)}{\left(1 - \frac{1}{p^{1+\beta+\gamma}}\right) \left(1 - \frac{1}{p^{1+\alpha+\delta}}\right)}$$

for $|\Re\alpha|, |\Re\beta| < 1/4$,

$$\frac{1}{\log T} \ll \Re\gamma, \Re\delta < 1/4, \Im\alpha, \Im\beta, \Im\gamma, \Im\delta \ll T^{1-\epsilon}.$$

Applications of the Ratios Conjecture

- Any integer can be written as the sum of three cubes (V. Wang, 2022)
- Compute the lower order terms for the pair correlation of the zeros of $\zeta(s)$, which were previously heuristically computed by Bogomolny and Keating.
- Compute mollified moments of $\zeta(s)$ or other L -functions
- Obtain conjectures for moments of $|\zeta'(\rho)|$
- Compute the one-level density of zeros in families of L -functions, for test functions whose Fourier transforms have any support.

Conjecture (Chowla's conjecture)

$L(1/2, \chi) \neq 0$ for any χ a Dirichlet character.

- Soundararajan: $\geq 87.5\%$ of $L(1/2, \chi_d) \neq 0$
- Ozluk-Snyder: $\geq 93.75\%$ of $L(1/2, \chi_d) \neq 0$ by computing the one-level density of zeros with support $(-2, 2)$ (GRH)
- The Ratios Conjecture $\Rightarrow 100\%$ of $L(1/2, \chi_d) \neq 0$

The Ratios Conjecture in Random Matrix Theory

One can compute ratios of characteristic polynomials in matrix ensembles:

- Conrey-Farmer-Zirnbauer
- Borodin-Strahov
- Conrey-Forrester-Snaith
- Bump-Gamburd
- Huckleberry-Puttmann-Zirnbauer

Theorem (Conrey-Farmer-Zirnbauer)

For $\Re \gamma_k > 0$, we have

$$\int_{U_{Sp(2N)}} \frac{\prod_{k=1}^K \Lambda_A(e^{-\alpha_k})}{\prod_{k=1}^K \Lambda_A(e^{-\gamma_k})} dA$$
$$= \sum_{\epsilon \in \{-1,1\}^K} e^{N \sum_{k=1}^K (\epsilon_k \alpha_k - \alpha_k)} \frac{\prod_{j \leq k \leq K} z(\epsilon_j \alpha_j + \epsilon_k \alpha_k) \prod_{q < r \leq K} z(\gamma_q + \gamma_r)}{\prod_{k=1}^K \prod_{q=1}^K z(\epsilon_k \alpha_k + \gamma_q)},$$

where $z(x) = (1 - e^{-x})^{-1}$.

Ideas of proof

- **Key Lemma** (adapted from work of Carneiro and Chandee)

$$\log \frac{1}{|\zeta(1/2 + \delta + it)|} \leq \Re \sum_{p \leq x} \frac{a(p)}{p^{\frac{1}{2} + \delta + it}} + \frac{\log T}{\log x} \log \frac{1}{1 - x^{-\delta}} + O(1).$$

- $|\zeta(1/2 + \delta + it)|^{-2k} \ll \left(\log \frac{1}{1 - x^{-2\delta}} \right)^{\frac{2k \log T}{\log x}} \exp \left(2k \Re \sum_{p \leq x} \frac{a(p)}{p^{\frac{1}{2} + \delta + it}} \right).$

- Let $\mathcal{P}_j(t) = \sum_{T^{\beta_{j-1}} \leq p \leq T^{\beta_j}} \frac{1}{p^{1/2 + \delta + it}},$

- $1 = T^{\beta_0}, T^{\beta_1}, \dots, T^{\beta_K}$

$$\beta_1 \asymp \frac{\log \log T}{\log T}, \beta_j = r^{j-1} \beta_1, r > 1$$

Stop when $\beta_K = \text{small constant}.$

Ideas of proof

- Assume $\delta \gg \frac{1}{(\log T)^{\frac{1}{2k}-\epsilon}}$.
- There are 3 possibilities for $t \in [0, T]$:
- ① $t \in \mathcal{S}_0$ "Exceptional Set"

$$|\Re \mathcal{P}_1(t)| > \beta_1^{-d}, \quad d = 1 - \epsilon.$$

$$\begin{aligned} \int_{\mathcal{S}_0} |\zeta(1/2 + \delta + it)|^{-2k} &\leq \int_0^T |\zeta(1/2 + \delta + it)|^{-2k} (\beta_1^d |\Re \mathcal{P}_1(t)|)^{s_0} dt \\ &\leq \beta_1^{d s_0} \left(\int_0^T |\zeta(1/2 + \delta + it)|^{-4k} \right)^{1/2} \left(\int_0^T |\Re \mathcal{P}_1(t)|^{2s_0} \right)^{1/2} \end{aligned}$$

$$\int_{\mathcal{S}_0} |\zeta(1/2 + \delta + it)|^{-2k} \leq \beta_1^{d_{\mathcal{S}_0}} \left(\int_0^T |\zeta(1/2 + \delta + it)|^{-4k} \right)^{1/2} \left(\int_0^T (|\Re \mathcal{P}_1(t)|)^{2s_0} \right)^{1/2}$$

- Use a priori bound for the first term; use pointwise bound

$$\frac{1}{|\zeta(1/2 + \delta + it)|} \ll \left(\frac{1}{1 - (\log T)^{-2\delta}} \right)^{\frac{\log T}{2 \log \log T}}.$$

- Compute moments of the sum over the primes. Need $\beta_0 s_0 \leq 1$.
- Contribution from exceptional set is $o(T)$.

Ideas of proof

- ② t is such that $|\Re\mathcal{P}_h(t)| \leq \beta_h^{-d}$, $h \leq j$, but

$$|\Re\mathcal{P}_{j+1}(T)| > \beta_{j+1}^{-d}.$$

Call this set \mathcal{T}_j . Let

$$E_\ell(t) = \sum_{s \leq \ell} \frac{t^s}{s!}.$$

If $t \leq \ell/e^2$, then

$$e^t \leq (1 + e^{-\ell/2})E_\ell(t).$$

- Since $|\Re\mathcal{P}_h(t)| \leq \beta_h^{-d}$, we can approximate

$$\exp(2k\Re\mathcal{P}_h(t)) \ll E_{(\beta_h)^{-d}}(2k\Re\mathcal{P}_h(t)).$$

Ideas of proof

- Use Key Lemma with $x = T^{\beta_j}$. We get that

$$\begin{aligned} & \int_{\mathcal{T}_j} |\zeta(1/2 + \delta + it)|^{-2k} \leq \exp\left(\frac{2k}{\beta_j} \log \frac{1}{1 - T^{-\beta_j \delta}}\right) \\ & \times \int_{\mathcal{T}_j} \exp\left(\sum_{p \leq T^{\beta_j}} \frac{a(p)}{p^{1/2 + \delta + it}}\right) dt \\ & \ll \exp\left(\frac{2k}{\beta_j} \log \frac{1}{1 - T^{-\beta_j \delta}}\right) \int_0^T \prod_{h \leq j} E_{\beta_h^{-d}}(2k \Re \mathcal{P}_h(t)) \\ & \times (\beta_{j+1}^d |\Re \mathcal{P}_{j+1}(t)|)^{s_{j+1}} dt \end{aligned}$$

- Can compute moments as long as

$$\sum_{h=0}^j \beta_h^{1-d} + s_{j+1} \beta_{j+1} < 1.$$

- Contribution is small.

Ideas of proof

- ③ t is such that $|\Re\mathcal{P}_h(t)| \leq \beta_h^{-d}$, $h \leq K$. Call this set \mathcal{T}_K .
- Use Key Lemma with $x = T^{\beta_K}$. We get that

$$\int_{\mathcal{T}_K} |\zeta(1/2 + \delta + it)|^{-2k} \leq \exp\left(\frac{2k}{\beta_K} \log \frac{1}{1 - T^{-\beta_K \delta}}\right) \\ \times \int_0^T \prod_{h \leq K} E_{\beta_h^{-d}}(2k \Re\mathcal{P}_h(t)) dt$$

- Can compute moments as long as

$$\sum_{h=0}^K \beta_h^{1-d} < 1.$$

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$$\frac{1}{T} \int_{\mathcal{T}_K} |\zeta(1/2 + \delta + it)|^{-2k} \ll (\log T)^{k^2 + \epsilon}.$$

Smaller shifts

- Perform an inductive argument. First step of the induction is similar to the previous case. Let

$$u = \frac{\log(1/\delta)}{\log \log T}.$$

Obtain

$$\frac{1}{T} \int_0^T |\zeta(1/2 + \delta + it)|^{-2k} \ll T^{ku(1 - \frac{1}{4ku-1})}.$$

- Second step of the induction. Obtain

$$\frac{1}{T} \int_0^T |\zeta(1/2 + \delta + it)|^{-2k} \ll T^{ku(1 - \frac{1}{4ku-1})(1 - \frac{1}{8ku-1})}.$$

- Repeat...

Thank you!