One-level density of zeros of Dirichlet L-functions over function fields

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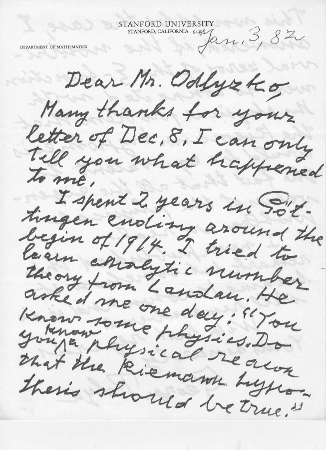
July 25, 2022

The Hilbert-Pólya Conjecture



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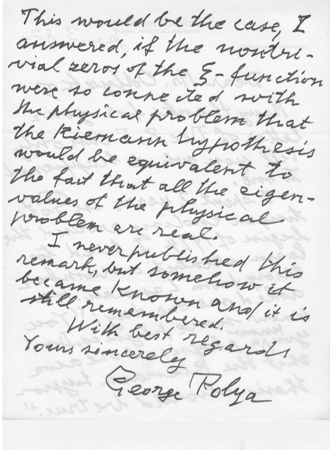
Jan. 3, 82

Dear Mr. Odlyzko,
Many thanks for your letter of Dec. 8. I can only tell you what happened to me.
I spent 2 years in Göttingen ending around the begin of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true."



This would be the case, I answered, if the nontrivial zeros of the Ξ-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.
I never published this remark, but somehow it became known and it is still remembered.
With best regards
Yours sincerely
George Pólya

"He [Landau] asked me one day: "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true." ...I answered, if the nontrivial zeros of the Xi-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.

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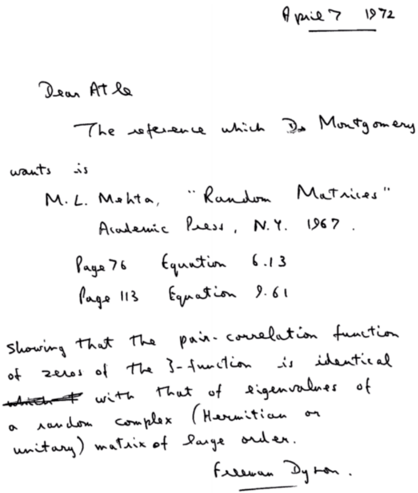
Dyson's Note to Montgomery



April 7 1972

Dear Atle

The reference which Dr Montgomery wants is

M. L. Mehta, "Random Matrices"
Academic Press, N.Y. 1967.

Page 76 Equation 6.13
Page 113 Equation 9.61

showing that the pair-correlation function of zeros of the ζ-function is identical ~~which~~ with that of eigenvalues of a random complex (Hermitian or unitary) matrix of large order.

Freeman Dyson.

Montgomery computed the pair correlation of zeros of the Riemann zeta function under GRH, and found the function $1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$. The famous physicist Freeman Dyson recognized it and pointed out that the eigenvalues of a random complex Hermitian or unitary matrix of large order have precisely the same distribution function [Mon73].
The note on the left proves it!

Katz and Sarnak's philosophy

Katz and Sarnak predicted that the statistics for the zeros in families of L-functions, in the limit when the conductor of the L-functions gets large, follow the distribution laws of classical random matrices

- pair correlation of zeros of zeta functions of curves of large genus over large finite fields satisfy the Montgomery law (1999)
- this holds for all families where analogue of Deligne's deep equidistribution theorem (1974, 80) applies
- one-level density of zeros of Dirichlet L-functions as the genus gets large should also follow Katz and Sarnaks's philosophy

Results over number fields

- Özlük and Snyder [OS93], one-level density of quadratic Dirichlet L-functions
  - more than 93.75% non-vanishing at $s = 1/2$
  - scaling density (symmetry type) associated to the group of unitary symplectic matrices
- Cho and Park [CP19], one-level density of cubic L-functions in the Kummer setting under GRH matches that predicted by a heuristic (Ratios Conjecture, [CFZ08])
- David and Güloğlu (2021), a thin family of L-functions with cubic characters over the Eisenstein field have a positive proportion of non-vanishing at $s = 1/2$ under GRH

From number fields to function fields

Over Number Fields	Over Function Fields
$\mathbb{Z}$	$\mathbb{F}_q[t]$
$\mathbb{Q}$	$\mathbb{F}_q(t)$
n positive integer	F monic polynomial
p prime	P irreducible (prime)
$\lvert n\rvert = \lvert\mathbb{Z}/n\mathbb{Z}\rvert$	$\lvert F\rvert_q = \lvert\mathbb{F}_q[t]/F\rvert = q^{\deg(F)}$
$L(s,\chi) = \sum_{n=1} \frac{\chi(n)}{n^s} = \prod_p (1 - \frac{\chi(p)}{p^s})^{-1}$	$L(s,\chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{\lvert f\rvert_q^s} = \prod_{P \text{ prime}} (1 - \chi(P)u^{d(P)})^{-1}$

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It is convenient to make the change of variable $u = q^{-s}$, and we define

$$\mathcal{L}_q(u,\chi) := L_q(s,\chi) = \sum_{f \in \mathcal{M}_q} \chi(f)u^{d(f)} = \prod_{\substack{P \in \mathcal{P}_q \\ P \nmid h}} \left(1 - \chi(P)u^{d(P)}\right)^{-1}.$$

What is one-level density over function fields?

One-level density of zeros is a statistic of low lying zeros of families of Dirichlet L-functions.
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- each character has a conductor $h(t)$, which is similar to the modulus of the character (if a character has a modulus P, then it has conductor P)
- we call the case when $q \equiv 1 \pmod{\ell}$ Kummer (containing roots of unity), and non-Kummer otherwise

A non-Kummer example

Test function $\phi(\theta)$: any real even trigonometric polynomial, for example $\frac{\sin(2\pi\theta)}{2\pi\theta}$. Let $\Phi(D\theta) = \phi(\theta)$ for some positive number D.
For example, the one-level density of order ℓ L-functions in the function field $\mathbb{F}_q[t]$ in the non-Kummer setting ($q \not\equiv 1 \pmod{\ell}$) is

$$\Sigma_{\ell}^{\mathrm{nK}}(\Phi, g) = \frac{1}{|\mathcal{H}|} \sum_{F \in \mathcal{H}} \sum_{j=1}^{\deg(h)-2} \Phi\left(\left(\deg(h) - 2\right) \theta_{j,F}\right). \tag{1}$$

- $\mathcal{H}$ denotes the family of L-functions and $|\mathcal{H}|$ denotes the size of the family
- the outer sum sums over the family
- the inner sum sums over the zeros

Results over function fields

- Rudnick [Rud10], one-level density of quadratic L-functions with test function whose Fourier transform is supported in (-2,2) has a symmetry type of symplectic matrices
- Bui and Florea [BF18] get the same result with extra lower order terms for the same family, showed a 94.27% nonvanishing
- We compute the one-level density of zeros of cubic and quartic Dirichlet L-functions in the Kummer setting, and cubic, quartic and sextic Dirichlet L-functions in the non-Kummer setting, and confirm the symmetry type of the family is unitary

Quartic non-Kummer results

Theorem (L.)

Let $\phi(\theta) = \sum_{|n|\leq N} \hat{\phi}(n)e(n\theta)$ *be any real, even trigonometric polynomial and* $\Phi\left(\frac{2g\theta}{3}\right) = \phi(\theta)$. *Let* $\Sigma_4^{\mathrm{nK}}(\Phi, g)$ *be the one-level density of quartic Dirichlet L-functions in the non-Kummer setting. We have that*

$$\Sigma_4^{\mathrm{nK}}(\Phi, g) = \hat{\Phi}(0) - \frac{3}{g}\sum_{1\leq n\leq N}\hat{\Phi}\left(\frac{3n}{2g}\right)q^{-n/2}$$
$$-\frac{3}{g}\sum_{1\leq n\leq N/\ell}\hat{\Phi}\left(\frac{6n}{g}\right)\sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\geq 1}}\frac{d(Q)}{|Q|_q^{2r}\left(1+|Q|_q^{-2/m_Q}\right)^{m_Q}} + O\left(q^{N/2}q^{-g/3}q^{\epsilon(N+g)}\right),$$

where $m_Q = \gcd(d(Q), 2)$.

Cubic non-Kummer results

Theorem (L.)

Let $\phi(\theta) = \sum_{|n| \leq N} \hat{\phi}(n) e(n\theta)$ *be any real, even trigonometric polynomial and* $\Phi\left(g\theta\right) = \phi(\theta)$. *Let* $\Sigma_3^{\mathrm{nK}}\left(\Phi, g\right)$ *be the one-level density of cubic Dirichlet L-functions in the non-Kummer setting. We have that*

$$\Sigma_3^{\mathrm{nK}}\left(\Phi, g\right) = \hat{\Phi}(0) - \frac{2}{g} \sum_{1 \leq n \leq N} \hat{\Phi}\left(\frac{n}{g}\right) q^{-n/2}$$

$$- \frac{2}{g} \sum_{1 \leq n \leq N/3} \hat{\Phi}\left(\frac{3n}{g}\right) \sum_{\substack{Q \in \mathcal{P}_{q,n/r} \\ r \geq 1}} \frac{\deg(Q)}{|Q|_q^{3r/2}\left(1 + |Q|_q^{-2/m_Q}\right)^{m_Q}} + O\left(q^{N/2} q^{-g/2} q^{\epsilon(N+g)}\right),$$

where $m_Q = \gcd(d(Q), 2)$.

Corollary (Symmetry type of the family)

Under the same condition as the theorem above, for $N < g$, we have in the cubic non-Kummer setting,

$$\lim_{g\to\infty} \Sigma_3^{\mathrm{nK}}(\Phi, g) = \int_{-\infty}^{\infty} \hat{\Phi}(y)\hat{W}_{U(g)}(y)dy + o(1).$$

Here $\hat{W}_{U(g)}(y) = \delta_0(y)$ denotes the one-level scaling density of the group of unitary matrices.

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Sketch of proof:

$$\lim_{g\to\infty} \Sigma_\ell^{\mathrm{nK}}(\Phi, g) = \hat{\Phi}(0),$$

since the sum over n and Q are $o(1)$ as $g \to \infty$. Thus

$$\hat{\Phi}(0) = \int_{-\infty}^{\infty} \hat{\Phi}(y)\hat{\mathcal{W}}_{U(g)}(y)dy = \int_{-\infty}^{\infty} \hat{\Phi}(y)\delta_0(y)dy.$$

This matches the philosophy of Katz and Sarnak.

Notations

The subscript q denotes polynomials are in $\mathbb{F}_q[t]$.

- $\mathcal{M}_{q,n}$ monic polynomials of degree n
- $\mathcal{P}_{q,n}$ monic irreducible polynomials of degree n
- $\mathcal{H}_{q,n}$ monic, squarefree polynomials of degree n
- $\mathcal{C}_{\ell}^{\mathrm{nK}}(g)$ the family of primitive order ℓ characters over $\mathbb{F}_q[t]$ of genus g in the non-Kummer setting

$\mathcal{M}_q$, $\mathcal{P}_q$, $\mathcal{H}_q$ are the corresponding sets without a degree restriction.

Primitive characters in the Kummer setting

Definition (ℓ^{th} residue symbol when ℓ divides $q-1$)

Let P be a monic irreducible polynomial and $f \in \mathbb{F}_q[t]$. The ℓ^{th} Jacobi symbol $\left(\frac{f}{P}\right)_\ell$ is the unique element of $\mathbb{F}_q^\times$ such that

$$f^{\frac{|P|-1}{\ell}} \equiv \left(\frac{f}{P}\right)_\ell \pmod{P}.$$

Definition (primitive order ℓ characters with conductor P)

Let Ω be a fixed isomorphism from the ℓ^{th} roots of unity $\mu_\ell \subseteq \mathbb{C}^\times$ to the ℓ^{th} roots of unity in $\mathbb{F}_q^\times$. We define $\chi_P(f) = 0$ if $P \mid f$, and otherwise

$$\chi_P(f) = \Omega^{-1}\left(\left(\frac{f}{P}\right)_\ell\right).$$

The Kummer setting

For a monic polynomial $H = P_1^{e_1} \cdots P_s^{e_s}$ with distinct primes P_i, we define

$$\chi_H = \chi_{P_1}^{e_1} \cdots \chi_{P_s}^{e_s},$$

where χ_H is a primitive character with $\chi_H^\ell = 1$ and conductor $F = P_1 \cdots P_s \iff$ all the e_i are natural numbers less than ℓ. For $\ell = 3$, e_i is either 1 or 2 as in [DFL19]. Grouping by the exponents, we can thus express χ_H as $\chi_{F_1}\chi_{F_2}^2 \cdots \chi_{F_{\ell-1}}^{\ell-1}$ where $H = F_1 F_2^2 \cdots F_{\ell-1}^{\ell-1}$, and the F_i are monic squarefree polynomials and pairwise coprime. Thus given a conductor $F = F_1 F_2 \cdots F_{\ell-1}$, we have the corresponding primitive character

$$\chi_{F_1}\chi_{F_2}^2 \cdots \chi_{F_{\ell-1}}^{\ell-1}.$$

The Kummer setting

For convenience, we restrict to the case of odd primitive characters where $\deg(H) \equiv 1 \pmod{\ell}$ and $d(F) = D(\ell) - 1$, where $D(\ell) = \frac{2g+2\ell-2}{\ell-1}$ given by the Riemann-Hurwitz formula [BSM19]. We have thus

$$\Sigma_\ell^{\mathrm{K}}(\Phi, g) = \frac{1}{\left|\mathcal{C}_\ell^{\mathrm{K}}(g)\right|} \sum_F^* \sum_{j=1}^{D(\ell)-2} \Phi\left((D(\ell)-2)\theta_{j,F}\right) =$$

$$\hat{\Phi}(0) - \frac{1}{|\mathcal{C}_\ell^{\mathrm{K}}(g)|(D(\ell)-2)} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D(\ell)-2}\right) \sum_{f\in\mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_F^* \left[\chi_F(f) + \overline{\chi_F(f)}\right], \tag{2}$$

where, for example, in the cubic case, the sum over F

$$\sum_F^* = \sum_{\substack{d_1+d_2=g+1 \\ d_1\equiv a \pmod 3}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ (F_1,f)=1}} \sum_{\substack{F_2\in\mathcal{H}_{q,d_2} \\ (F_2,F_1f)=1}} \left[\chi_F(f) + \overline{\chi_F(f)}\right].$$

Primitive characters of order ℓ =3, 4 and 6 in the non-Kummer setting

When $\ell \nmid q - 1$

- $q \equiv -1 \pmod{\ell}$ since q is a prime power, and $q^2 \equiv 1 \pmod{\ell}$
- define the ℓ^{th} Jacobi symbol and order ℓ primitive characters analogously for $\mathbb{F}_{q^2}$
- primitive order ℓ characters have conductors of even degree since χ_P is an order ℓ character $\implies \ell \mid q^{\deg(P)} - 1$ and $2 \mid \deg(P)$
- these primitive characters are the restrictions of some characters of $\mathbb{F}_{q^2}[t]$ down to $\mathbb{F}_q[t]$, [DFL19], [BSM19], [BY10]

Primitive characters of order ℓ =3, 4 and 6 in the non-Kummer setting

Let P be a prime in $\mathbb{F}_q[t]$ of even degree

- P splits into 2 primes $\pi_1\pi_2$ in $\mathbb{F}_{q^2}[t]$
- the restriction $\{\chi_{\pi_i}\}|_{\mathbb{F}_q[t]}$ is the set of primitive characters $\{\chi_P^{e_i}\}$ for $(e_i, \ell) = 1$
- primitive order ℓ characters χ_F where χ_F^d stays primitive for proper nontrivial divisor d of ℓ are given by $F \in \mathbb{F}_{q^2}[t]$, F square-free and not divisible by a prime of $\mathbb{F}_q[t]$
- $\deg(F) = D(\ell)/2$ for conductor $F \in \mathbb{F}_{q^2}[t]$

The non-Kummer setting

We start with the familiar one-level density sum

$$\Sigma_{\ell}^{\mathrm{nK}}(\Phi, g) = \frac{1}{\left|\mathcal{C}_{\ell}^{\mathrm{nK}}(g)\right|} \sum_{F \in \mathcal{C}_{\ell}^{\mathrm{nK}}(g)} \sum_{j=1}^{D(\ell)-2} \Phi\left(\left(D(\ell)-2\right)\theta_{j,F}\right). \tag{3}$$

Then applying the discussion above on the primitive characters (and use an explicit formula)

$$\Sigma_{\ell}^{\mathrm{nK}}(\Phi, g) = \hat{\Phi}(0) - \frac{2}{D(\ell)-2} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{n}{D(\ell)-2}\right) q^{-n/2}$$
$$- \frac{1}{\left|\mathcal{C}_{\ell}^{\mathrm{nK}}(g)\right| \left(D(\ell)-2\right)} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{n}{D(\ell)-2}\right) \sum_{f \in \mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{\substack{F \in \mathcal{H}_{q^2, D(\ell)/2} \\ P \mid F \Rightarrow P \notin \mathbb{F}_q[t]}} \left[\chi_F(f) + \overline{\chi_F(f)}\right]. \tag{4}$$

One-level density in the cubic non-Kummer case

To compute the cubic case, we start with

- $$\Sigma_3^{\mathrm{nK}}(\Phi, g) = \frac{1}{\left|\mathcal{C}_3^{\mathrm{nK}}(g)\right|} \sum_{\substack{F \in \mathcal{H}_{q^2,(g+2)/2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[t]}} \sum_{j=1}^{g} \Phi\left(g\theta_{j,F}\right). \tag{5}$$

One-level density in the cubic non-Kummer case

To compute the cubic case, we start with

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  use an explicit formula to write the sum over zeros to the sum over primes, we have that

- $$\begin{aligned}\sum_{j=1}^{g} \Phi(g\theta_{j,F}) =& \hat{\Phi}(0) - \frac{2}{g} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{g}\right) q^{-n/2} \\ & - \frac{1}{g} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{g}\right) \sum_{f\in\mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \left[\chi_F(f) + \overline{\chi_F(f)}\right],\end{aligned}$$

The cubic non-Kummer case

The last term above becomes

$$\mathcal{A}_3^{\mathrm{nK}}(\Phi, g) = \frac{1}{\left|\mathcal{C}_3^{\mathrm{nK}}(g)\right| g} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{n}{g}\right) \sum_{f \in \mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_{\substack{F \in \mathcal{H}_{q^2,(g+2)/2} \\ P|F \Rightarrow P \notin \mathbb{F}_q[t]}} \left[\chi_F(f) + \overline{\chi_F(f)}\right], \tag{6}$$

where the main term contribution comes from when f is a cube.
Thus the one-level density we want to compute is

$$\Sigma_3^{\mathrm{nK}}(\Phi, g) = \hat{\Phi}(0) - \mathcal{A}_3^{\mathrm{nK}}(\Phi, g) - \frac{2}{g} \sum_{1 \le n \le N} \hat{\Phi}\left(\frac{n}{g}\right) q^{-n/2}. \tag{7}$$

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- main term: square-free sieve, generating series, Perron's formula

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- main term: square-free sieve, generating series, Perron's formula
- error term: similar to the main term, Lindelöf hypothesis bounds

Thank you!

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One-level density result for quadratic L-functions

Theorem (Rudnick 10, Bui and Florea 16)

$$\frac{1}{|\mathcal{H}_{2g+1}|}\sum_{D\in\mathcal{H}_{2g+1}}\sum_{j=1}^{2g}\Phi(2g\theta_j)=\int_{-\infty}^{\infty}\hat{\Phi}(y)\hat{W}_{Sp(2g)}(y)dy+o(1)$$

as $g\to\infty$ for any fixed, even Φ with the support of $\hat{\Phi}$ in $(-2,2)$, where $\hat{W}_{Sp(2g)}(y)=\delta_0(y)-\frac{1}{2}\eta(y)$ with η being the characteristic function of the interval $[-1,1]$.

One-level density in the Kummer setting

Lemma (main term)

Let q be a prime power coprime to 6 and $f \in \mathcal{M}_q$. In the Kummer setting we have

$$\sum_{\substack{\chi \text{ primitive}, \ \chi^\ell=1 \\ genus(\chi)=g \\ \chi|_{\mathbb{F}_q^\times}=\chi_\ell}} \chi(f) = \sum_{\substack{d_1+\cdots+d_{\ell-1}=D(\ell)-1 \\ d_1+\cdots+(\ell-1)d_{\ell-1}\equiv 1 \pmod{\ell}}} \sum_{\substack{F_1\in\mathcal{H}_{q,d_1} \\ (F_1,f)=1}} \chi_{F_1}(f) \cdots \sum_{\substack{F_{\ell-1}\in\mathcal{H}_{q,d_{\ell-1}} \\ (F_{\ell-1},F_1F_2\cdots f)=1}} \chi_{F_{\ell-1}}^{\ell-1}(f).$$

$$\Sigma_\ell^{\mathrm{K}}(\Phi, g) = \hat{\Phi}(0) - \frac{1}{|\mathcal{C}_\ell^{\mathrm{K}}(g)|(D(\ell)-2)} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D(\ell)-2}\right) \sum_{f\in\mathcal{M}_{q,n}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_F{}^* \left[\chi_F(f) + \overline{\chi_F(f)}\right]. \quad (8)$$

We decompose the latter term as the sum $M_\ell^{\mathrm{K}}(\Phi, D(\ell)-2) + E_\ell^{\mathrm{K}}(\Phi, D(\ell)-2)$, where the main term comes from when f is an ℓ^{th} power.

$$M_\ell^{\mathrm{K}}(\Phi, D(\ell)-2) = \frac{1}{|\mathcal{C}_\ell^{\mathrm{K}}(g)|(D(\ell)-2)} \sum_{1\le n\le N/\ell} \hat{\Phi}\left(\frac{\ell n}{D(\ell)-2}\right) \sum_{\substack{Q\in\mathcal{P}_{q,n/r}\\ r\ge 1}} \frac{d(Q)}{|Q|^{\ell r/2}} \sum_F{}^{*} \chi_F\left(Q^{\ell r}\right) + \overline{\chi_F\left(Q^{\ell r}\right)}, \tag{9}$$

and the non-ℓ^{th} power contribution is

$$E_\ell^{\mathrm{K}}(\Phi, D(\ell)-2) = \frac{2}{|\mathcal{C}_\ell^{\mathrm{K}}(g)|(D(\ell)-2)} \sum_{1\le n\le N} \hat{\Phi}\left(\frac{n}{D(\ell)-2}\right) \sum_{\substack{f\in\mathcal{M}_{q,n}\\ f \text{ non-}\ell^{th} \text{ power}}} \frac{\Lambda(f)}{|f|^{1/2}} \sum_F{}^{*} \chi_F(f) + \overline{\chi_F(f)}. \tag{10}$$

non-Kummer lemma

Lemma

Let f be a monic polynomial in $\mathbb{F}_q[t]$. For $\ell = 3$,

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} \chi(f) = \sum_{\substack{F \in \mathcal{H}_{q^2, D(3)/2} \\ P|F \implies P \notin \mathbb{F}_q[t]}} \chi_F(f).$$

For $\ell = 4$, we have

$$\sum_{\substack{\chi \text{ primitive order } 4 \\ \text{genus}(\chi)=g \\ \chi^2 \text{primitive}}} \chi(f) = \sum_{\substack{F \in \mathcal{H}_{q^2, D(4)/2} \\ P|F \implies P \notin \mathbb{F}_q[t]}} \chi_F(f).$$

We have a similar result for $\ell = 6$ which we omit here due to length.