Specification and the measure of maximal entropy

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June 23, 2020

A finite set (alphabet) $\quad A^{\mathbb{N}} = \{x_1 x_2 \cdots : x_n \in A\}$ $\quad \sigma : A^{\mathbb{N}} \circlearrowleft$ $\quad \sigma(x)_n = x_{n+1}$

$d(x,y) = e^{-n(x,y)}$ $\quad n(x,y) = \min\{n : x_n \neq y_n\}$

$X \subset A^{\mathbb{N}}$ closed & $\sigma(X) = X$ $\quad$ shift space

Example: $A = \{1, \dots, d\}$ $\quad T$ $d \times d$ matrix $\quad T_{ij} \in \{0,1\}$

$i \to j$ if $T_{ij} = 1$ $\quad i \not\to j$ if $T_{ij} = 0$ $\quad$ (TMS)

$X = \{x \in A^{\mathbb{N}} : x_n \to x_{n+1} \ \forall n\}$

mixing/primitive: $\exists N \in \mathbb{N}$ s.t. $(T^N)_{ij} > 0 \ \forall i,j$

$\mathcal{M} = \{\text{Borel prob meas. on } X\}$

$\mathcal{M}_\sigma = \{\mu \in \mathcal{M} : \sigma_* \mu = \mu\}$ $\quad \sigma_* \mu = \mu \circ \sigma^{-1}$

$\mathcal{M}_\sigma^e = \{\mu \in \mathcal{M}_\sigma : \mu \text{ ergodic}\}$

$h_{top}(X) = \sup\{h(\mu) : \mu \in \mathcal{M}_\sigma\}$

$h(\mu) = h_{top}(X)$: μ MME

$\boxed{\text{Thm}}$ specification

$\Rightarrow$ unique MME

Generalizations:

1. non-symbolic (expansivity)
2. equilibrium states $\left(h(\mu) + \int \varphi \, d\mu\right)$
3. ctbl alphabet / weaker spec.
4. mixing, K, Bernoulli, LDP, CLT, EDC

$w \in A^n$ cylinder $[w] = X \cap wX$ $\mathcal{L}_n = \{ w \in A^n : [w] \neq \emptyset \}$ $\mathcal{L} = \bigcup_{n \geq 0} \mathcal{L}_n$ language

X has specification if $\exists\, \tau \in \mathbb{N}_0$ s.t. $\forall\, v, w \in \mathcal{L}$ $\exists\, u \in \mathcal{L}_\tau$ s.t. $vuw \in \mathcal{L}$

Mixing TMS $\rightarrow$ specification.

Topological entropy /

$\mathcal{L}_{m+n} \subset \mathcal{L}_m \mathcal{L}_n \rightarrow \# \mathcal{L}_{m+n} \leq \# \mathcal{L}_m \# \mathcal{L}_n$ (complexity) $c_n = \log \# \mathcal{L}_n$ $\boxed{c_{m+n} \leq c_m + c_n}$ ⊛

Fekete's Lemma: ⊛ $\Rightarrow \lim_{n \to \infty} \frac{c_n}{n} = \inf_n \frac{c_n}{n}$ $\therefore$ $\lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{L}_n$ exists. Call it $h_{top}(X)$.

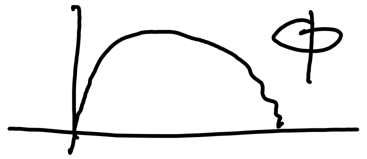
NB $\frac{c_n}{n} \geq h_{top}(X)$ $\forall\, n$ $\therefore$ $\log \# \mathcal{L}_n \geq n h_{top}(X)$ $\therefore$ $\# \mathcal{L}_n \geq e^{n h_{top}(X)}$

(strengthen "subexp." to "uniform")

Measure-theoretic entropy /

$p \in (0,1]$ $I(p) = -\log p$ $I(pq) = I(p) + I(q)$

$\phi(p) = p\, I(p)$ $\Delta_N = \{ (p_1, \ldots, p_N) = \bar{p} : p_i \geq 0, \sum p_i \leq 1 \}$



concave

$H(\bar{p}) = \sum_{i=1}^{N} \phi(p_i)$

$H_{m+n}(\mu) \leq H_m(\mu) + H_n(\mu)$

$\mu \in \mathcal{M}_\sigma$ $\mu(w) = \mu([w])$ $H_n(\mu) = \sum_{w \in \mathcal{L}_n} \phi(\mu(w))$ $\lim_{n \to \infty} \frac{1}{n} H_n(\mu) =: h(\mu)$

$\frac{H_n}{n} \geq h(\mu) \Rightarrow H_n(\mu) \geq n h(\mu)$

Var princ /

$H(\bar{p}) \leq \log N$, $=$ iff $p_i = \frac{1}{N}$ $\forall\, i$ $\therefore$ $H_n(\mu) \leq \log \# \mathcal{L}_n = c_n$ $\therefore$ $h(\mu) \leq h_{top}(X)$.

Def: $\mu \in \mathcal{M}_\sigma$ is Gibbs if $\exists\ c, C, h > 0$ s.t. $\forall\ n \in \mathbb{N}$, $w \in \mathcal{L}_n$, $ce^{-nh} \le \mu(w) \le Ce^{-nh}$

$\boxed{\text{Thm 1}}$ If μ is Gibbs then $h = h_{top}(X) = h(\mu)$.

$I(p) = -\log p$

If μ is ergodic Gibbs then $\forall\ \nu \ne \mu$, $h(\nu) < h$.

$\Phi(p) = -p\log p = pI(p)$

$\boxed{\text{Thm 2}}$ If X has spec, then $\exists$ ergodic Gibbs meas.

Pf of Thm 1. Lower bd $\Rightarrow$ $1 = \mu(X) \ge ce^{-nh} \# \mathcal{L}_n \Rightarrow \# \mathcal{L}_n \le c^{-1} e^{nh} \Rightarrow h_{top}(X) \le h$.

Upper bd $\Rightarrow$ $H_n(\mu) = \sum_{w \in \mathcal{L}_n} \mu(w) I(\mu(w)) \ge \sum_{w \in \mathcal{L}_n} \mu(w) I(Ce^{-nh}) = I(Ce^{-nh})$

$= nh - \log C \quad \therefore h(\mu) \ge h$.

$h_{top}(X) \le h \le h(\mu) \le h_{top}(X) \quad \therefore$ all $=$.

$\boxed{\text{Lem}}$ Given $\bar{p}, \bar{q} \in \Delta_N$ & $s, t \ge 0$, $s+t=1$, we have

$$sH(\bar{p}) + tH(\bar{q}) \le H(s\bar{p} + t\bar{q}) \le sH(\bar{p}) + tH(\bar{q}) + \log 2$$

$\boxed{\text{Cor}}$ $\nu_1, \nu_2 \in \mathcal{M}_\sigma$ $\quad sH_n(\nu_1) + tH_n(\nu_2) \le H_n(s\nu_1 + t\nu_2) \le sH_n(\nu_1) + tH_n(\nu_2) + \log 2$

$$\therefore \quad h(s\nu_1 + t\nu_2) = sh(\nu_1) + t(\nu_2)$$

Given $\nu \in \mathcal{M}_\sigma$, $\nu \ne \mu$, Leb. decomp. $\nu = s\nu_1 + t\nu_2$, $\nu_1 \perp \mu$, $\nu_2 \ll \mu$ $\quad$ (μ erg $\Rightarrow \nu_2 = \mu$) $\therefore s > 0$

$h(\nu) = sh(\nu_1) + th(\mu)$ $\quad$ This is $< h$ iff $h(\nu_1) < h$.

$sH(\bar{p})+tH(\bar{q}) \leq H(s\bar{p}+t\bar{q}) \leq sH(\bar{p})+tH(\bar{q})+\log 2$

$\nu|_{\mathcal{L}_n} = \nu(\mathcal{D}_n)\,\nu|_{\mathcal{D}_n} + \nu(\mathcal{D}_n^c)\,\nu|_{\mathcal{D}_n^c}$

Suppose $\nu \in \mathcal{M}_\sigma$, $\nu \perp \mu$. Then $\exists\, D \subset X$ s.t. $\nu(D)=0$ & $\mu(D)=1$.

$\therefore \exists\, \mathcal{D}_n \subset \mathcal{L}_n$ s.t. $\nu(\mathcal{D}_n) \to 0$ & $\mu(\mathcal{D}_n) \to 1$.

$$
\begin{aligned}
nh(\nu) &\leq H_n(\nu) = H_n\left(\nu(\mathcal{D}_n)\,\nu|_{\mathcal{D}_n} + \nu(\mathcal{D}_n^c)\,\nu|_{\mathcal{D}_n^c}\right) \\
&\leq \nu(\mathcal{D}_n)\,H_n(\nu|_{\mathcal{D}_n}) + \nu(\mathcal{D}_n^c)\,H_n(\nu|_{\mathcal{D}_n^c}) + \log 2 \\
&\leq \nu(\mathcal{D}_n)\log(c^{-1}e^{nh}\mu(\mathcal{D}_n)) + \nu(\mathcal{D}_n^c)\log(c^{-1}e^{nh}\mu(\mathcal{D}_n^c)) + \log 2 \\
&= \underbrace{\log 2 + \log(c^{-1}e^{nh})}_{=\log\left(\frac{2}{c}\right)+nh} + \nu(\mathcal{D}_n)\log\mu(\mathcal{D}_n) + \nu(\mathcal{D}_n^c)\log\mu(\mathcal{D}_n^c)
\end{aligned}
$$

$$H_n(\nu|_{\mathcal{D}_n}) \leq \log \#\mathcal{D}_n \leq \log(c^{-1}e^{nh}\mu(\mathcal{D}_n))$$

lower Gibbs bd

$$n(h(\nu)-h) \leq \log\left(\frac{2}{c}\right) + \underbrace{\nu(\mathcal{D}_n)\log\mu(\mathcal{D}_n)}_{\to 0} + \underbrace{\overset{\to 1}{\nu(\mathcal{D}_n^c)}\,\overset{\to 0}{\log\mu(\mathcal{D}_n^c)}}_{\to -\infty}$$

$\therefore \not\to \infty$

$\therefore h(\nu) < h.$ ∎

$\boxed{\text{spec} \Longrightarrow \exists \text{ ergodic Gibbs}}$

$h = h_{top}(X)$.

$b_n = -\log \# \mathcal{L}_{n+\tau}$

① unif. counting bds

$\# \mathcal{L}_n \geq e^{nh}$

$v \in \mathcal{L}_m \quad w \in \mathcal{L}_n \rightarrow vuw \in \mathcal{L}_{m+n+\tau}$ $\quad \# \mathcal{L}_{m+n+\tau} \geq \# \mathcal{L}_m \# \mathcal{L}_n$

$\# \mathcal{L}_{k(n+\tau)} \geq (\# \mathcal{L}_n)^k$ $\quad \frac{1}{k(n+\tau)} \log \# \mathcal{L}_{k(n+\tau)} \geq \frac{1}{n+\tau} \log \# \mathcal{L}_n$

$\to h$

$\# \mathcal{L}_n \leq e^{\tau h} e^{nh}$

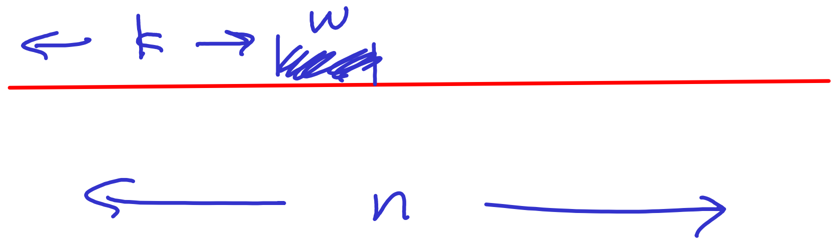
$$\boxed{e^{nh} \leq \# \mathcal{L}_n \leq e^{\tau h} e^{nh}}$$

② Gibbs

$\nu_n \in \mathcal{M}$ s.t. $\nu_n(w) = \frac{1}{\# \mathcal{L}_n} \quad \forall\, w \in \mathcal{L}_n$

$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \sigma_*^k \nu_n$ $\quad \mu_{n_j} \longrightarrow \mu$, $\mu \in \mathcal{M}_\sigma$

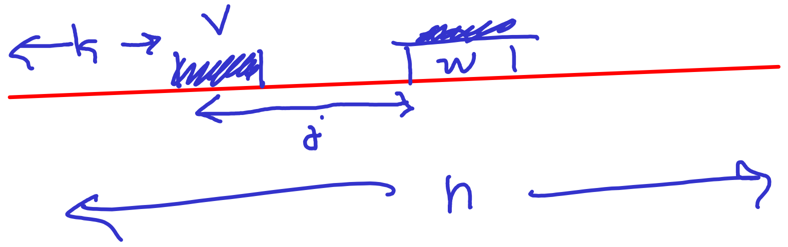
Estimate $\sigma_*^k \nu_n(w) = \nu_n(\sigma^{-k}[w]) = \frac{\#\{uwv \in \mathcal{L}_n : |u| = k\}}{\# \mathcal{L}_n}$



$$e^{-3\tau h} e^{-|w|h} \leq \frac{\# \mathcal{L}_{k-\tau} \# \mathcal{L}_{n-k-|w|-\tau}}{\# \mathcal{L}_n} \leq \sigma_*^k \nu_n(w) \leq \frac{\# \mathcal{L}_k \# \mathcal{L}_{n-k-|w|}}{\# \mathcal{L}_n} \leq e^{2\tau h} e^{-|w|h}$$

$$e^{-3\tau h} e^{-|w|h} \leq \mu(w) \leq e^{2\tau h} e^{-|w|h}$$

③ ergodic



$V, W \subset X$

Est. $\mu([v] \cap \sigma^{-j}[w])$ $\leq e^{9\tau h} \mu(v) \mu(w)$, $\geq e^{-9\tau h} \mu(v) \mu(w)$ $\quad \forall\, j \geq |v|$

$$\overline{\lim_{j \to \infty}}\, \mu(V \cap \sigma^{-j} W) = e^{\pm 9\tau h} \mu(V) \mu(W)$$

E inv. $\quad V = E \quad W = E^c$

$\mu(E)\mu(E^c) = 0$

$\therefore$ ergodic.