Limitations to equidistribution in arithmetic progressions

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Equidistribution in Arithmetic progressions

Let $\pi(x) = \#\{p \leq x : p\text{-prime}\}$,
and $\pi(x; q, a) = \#\{p \leq x : p\text{-prime}, p \equiv a \pmod{q}\}$.

- For $(a, q) = 1$, as $x \to \infty$,

$$\pi(x; q, a) \sim \frac{1}{\phi(q)} \pi(x).$$

- Denote

$$\Delta_\pi(x; q, a) := \pi(x; q, a) - \frac{\pi(x)}{\phi(q)}.$$

- **Bombieri-Vinogradov Theorem**: Given any $A > 0$,

$$\sum_{q \leq Q} \max_{(a,q)=1} \max_{y \leq x} \left| \Delta_\pi(x; q, a) \right| \ll_A \frac{x}{(\log x)^A}$$

holds for $Q = \frac{x^{1/2}}{(\log x)^B}$, for some $B = B(A) > 0$.

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Equidistribution seems to fail for large moduli

$\mathbf{EH}_\Lambda(Q)$: Given any $A > 0$,

$$\sum_{q \le Q} \max_{(a,q)=1} \max_{y \le x} \left| \sum_{\substack{n \le y \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{y}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}.$$

- **Elliott and Halberstam** conjectured that $\mathbf{EH}_\Lambda(Q)$ holds for $Q = x^\alpha/(\log x)^B$ for any $\alpha \le 1$.
- Call α to be the **level of distribution**.
- The case $\alpha = 1$ was disproved by Friedlander-Granville.

Theorem (Friedlander-Granville, 1989)

Fix $B > 1$. There exist arbitrarily large values of a and x for which

$$\sum_{\substack{q \le x/(\log x)^B \\ (a,q)=1}} \left| \sum_{\substack{n \le x \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{x}{\phi(q)} \right| \gg_B x.$$

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Equidistribution for a general function

Bombieri-Vinogradov type results: Can we expect equidistribution for other arithmetical functions?

Siebert and Wolke constructed a class of multiplicative functions f satisfying certain growth conditions, such that the equidistribution result

$$\sum_{q \leq x^{1/2}/(\log x)^B} \max_{(a,q)=1} \max_{y \leq x} \left| \Delta_f(y;q,a) \right| \ll \frac{x}{(\log x)^A}$$

holds, where

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"Disjunction" results

Granville-Soundararajan: At least one of the two following assumptions holds:

1. There is a discrepancy in the distribution in AP to a small modulus, i.e.,
$$\left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f(n) - \frac{1}{\phi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} f(n) \right|$$
is large for $q \leq x^{2/3}$.
2. There is a discrepancy in the distribution over large intervals, i.e.,
$$\left| \sum_{y < n \leq y+h} f(n) - \frac{h}{x} \sum_{\substack{n \leq x \\ (n,q)=1}} f(n) \right|$$
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Motivation

Let $\tau(n) = \#$divisors of n, then what can we say about

$$\Delta_\tau(x; q, a) = \sum_{\substack{n \le y \\ n \equiv a \pmod q}} \tau(n) - \frac{1}{\phi(q)} \sum_{\substack{n \le y \\ (n,q)=1}} \tau(n)?$$

Bombieri-Vinogradov type result: level of distribution $\theta = 2/3$ due to Hooley, Linnik and Selberg.

There are results due to Fouvry - Iwaniec, Banks - Heath-Brown - Shparlinski, Blomer, etc in the direction to push θ beyond $2/3$ by averaging over restricted moduli q.

Irving derived an upper bound for one particular $\Delta_\tau(x; q, a)$ for q as large as $x^{55/82}$ provided q is sufficiently smooth.

Friedlander-Granville type result: Open.

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Our results

- Friedlander-Granville type results for a family of functions satisfying certain hypotheses.
- Friedlander-Granville type of results for short arithmetic progressions.
- Applications to
  1. primes in short arithmetic progressions
  2. Beatty primes
  3. restricted divisor function defined as

$$\tau_z(n) = \begin{cases} \tau(n) & \text{if } P^-(n)^1 \geq z, \\ 0 & \text{otherwise.} \end{cases}$$

  4. $\phi(x, z) = \#\{n \leq x : P^-(n) > z\}$.

[1] $P^-(n) :=$ the smallest prime $p|n$

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Application to prime numbers

F-G type result for short arithmetic progressions:

Corollary 1 (S., Vatwani)

Fix $A > 1$. There exist arbitrarily large values of a, x and $h = h(x)$ in the range $x^{7/12} \leq h \leq x$, such that

$$\sum_{\substack{q \leq \frac{h}{(\log h)^A} \\ (q,a)=1}} \left| \sum_{\substack{x < n \leq x+h \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{h}{\phi(q)} \right| \gg_A \frac{h}{\log \log x}.$$

Application to Beatty primes

Given two real numbers α and β, the corresponding **Beatty sequence** is defined as

$$\mathcal{B}_{\alpha,\beta} = (\lfloor \alpha n + \beta \rfloor)_n.$$

Corollary 2(S., Vatwani)

Let $\alpha > 0$ be an irrational number of finite type[a] and let $\beta \in \mathbb{R}$. Fix $A \geq 1$. There are arbitrary large values of a and x for which we have

$$\sum_{\substack{q \leq x/(\log x)^A \\ (q,a)=1}} \left| \Delta_{\mathcal{B}_{\alpha,\beta}}(x; q, a) \right| \gg_A x.$$

[a] Say that α is of finite type if $\sup \{\rho \in \mathbb{R} : \liminf_{n\to\infty} n^{\rho} \|n\gamma\| = 0\} < \infty$, where $\|x\|$ denotes the distance of x from the nearest integer.

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Application to restricted divisor function

Corollary 3(S., Vatwani)

Fix $A > 1$. Let z be sufficiently large and $P = P(z) := \prod_{p<z} p$.

(i) Let $c_0 \geq 2$ and $\log z \ll D \leq z$. There exist arbitrarily large values of a, x satisfying $z \log z \ll \log x \ll z^{c_0}$, for which

$$\sum_{\substack{q \leq \frac{x}{(\log x)^A} \\ (q,a)=1}} \left| \Delta_{\tau_z}(x; q, a) \right| \gg_A \frac{x \log_2 x}{\log z}.$$

(ii) Let $\epsilon > 0$, $B > 1$ and $c_0 \geq 2$. There exist arbitrarily large values of a, x and $h = h(x)$, satisfying $z^{1+\frac{1}{B}} \ll \log x \ll z^{c_0}$ and $x^{1/2+\epsilon} \leq h(x) \leq o(x)$, for which

$$\sum_{\substack{q \leq \frac{h}{(\log h)^A} \\ (q,a)=1}} \left| \sum_{\substack{x<n\leq x+h \\ n \equiv a \ (\mathrm{mod}\ q)}} \tau_z(n) - \frac{1}{\phi(q)} \sum_{\substack{x<n\leq x+h \\ (n,q)=1}} \tau_z(n) \right| \gg_A \log\left(\frac{z^{c_0}}{\log x}\right) \frac{h \log_2 x}{(\log z)^2}.$$

GENERAL RESULT

A general class of functions

We will work with f satisfying certain hypotheses. Denote

$$f_z(n) := \begin{cases} f(n) & \text{if } P^-(n) \geq z, \\ 0 & \text{otherwise.} \end{cases}$$

Also, let $F(x,z) = \sum_{n \leq x} f_z(n)$ and $P = P(z) = \prod_{p<z} p$.

H1. $f(n) \ll \tau_k(n)$ for some $k \geq 1$.

H2. Let q free of prime factors below $\log q$. There exists a constant $\kappa_f \geq 0$ such that

$$\sum_{\substack{n \leq x \\ (n,q)=1}} f_z(n) = \left(1 + O\left(\frac{1}{\log z}\right)\right)\left(\frac{\phi(q)}{q}\right)^{\kappa_f} F(x,z),$$

uniformly in the range $x \geq P(z)^D$, and $P^{\frac{D}{2}} < q \leq \frac{P^D}{z \log z}$, where D satisfies $c \log z \leq D \leq z$ for some constant $c > 0$.

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A general class of functions (contd)

H3. (**Equidistribution over small moduli**) There exist arbitrarily large values of z for which

$$\sum_{\substack{x<n\leq x+h \\ n\equiv a \pmod{P}}} f_z(n) = \left(1 + O\left(\frac{1}{\log z}\right)\right)\frac{1}{\phi(P)}\big(F(x+h,z) - F(x,z)\big),$$

in the ranges $(a,P)=1$, $x \geq P^D$, $\frac{x}{2} \leq h \leq x$, where $c\log z \leq D \leq z$ for some constant $c > 0$.

(For primes, a renowned result due to Gallagher)

H4. (**Long average vs short average**) We have

$$\frac{1}{h}\sum_{X<n\leq X+h} f_z(n) = \left(1 + O\left(\frac{1}{\log z}\right)\right)\frac{1}{x}\sum_{n\leq x} f_z(n),$$

for $x \leq X \leq 2x$, $\frac{x}{2} \leq h \leq x$ and $x \geq P^D$, where D satisfies $c\log z \leq D \leq z$ for some constant $c > 0$.

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A general class of functions (contd)

Finally, we hypothesize a lower bound on the average order of f_z:

H5. Let k be as in **H1**. We have

$$\frac{1}{x}\big(F(2x,z) - F(x,z)\big) \gg \begin{cases} 1/\log z & \text{if } k = 1 \\ \exp\left(-\epsilon \frac{\log x}{\log_2 x}\right) \text{ for any } \epsilon > 0 & \text{if } k \geq 2, \end{cases}$$

in the range $x \geq P^D$, where D satisfies $c \log z \leq D \leq z$ for some constant $c > 0$.

Examples

- $f = \tau_k$: satisfies **H1** to **H5**:
- $f = 1$, so $F(x,z) = \Phi(x,z)$: satisfies **H1** to **H5**:
- $f = \Lambda$: satisfies **H2** to **H5**:

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Theorem A (S., Vatwani)

Fix $C > 1, c_0 \geq 2$. Assume that f satisfies **H1**-**H5**. Let z be sufficiently large. Then we have the following.
There exist arbitrarily large values of a and x satisfying

$$z \log z \ll \log x \leq z^{c_0}$$

such that

$$\sum_{\substack{q < \frac{x}{(\log x)^C} \\ (q,a)=1}} \left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} f_z(n) - \frac{1}{\phi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} f_z(n) \right| \gg_C \frac{(\log \log x)^2}{\log x} F(x, z).$$

Proof of Theorem A

Consider the following **Maier** matrix, having U columns and V rows.

$$M = \begin{bmatrix} (V+1)P+q & (V+1)P+2q & \dots & (V+1)P+Uq \\ \vdots & \vdots & \ddots & \vdots \\ (2V-1)P+q & (2V-1)P+2q & \dots & (2V-1)P+Uq \\ 2VP+q & 2VP+2q & \dots & 2VP+Uq \end{bmatrix}.$$

Let $f_z(M)$ be the matrix obtained by applying f_z to each entry.
Let Σ_M denote the sum of all the entries of $f_z(M)$.

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Counting Column-wise:

$$\Sigma_M = UV\left(\frac{1}{VP}F(VP,z)\right)\left(e^{\gamma}\omega\left(\frac{\log U}{\log z}\right)+O\left(\frac{1}{\log z}\right)\right).$$

Counting Row-wise:

$$\Sigma_M = \left(1+O\left(\frac{1}{\log z}\right)\right)UV\left(\frac{1}{VP}(F(VP,z))\right)+\Delta_q,$$

where,

$$\Delta_q := \sum_{\substack{V<r\leq 2V\\(r,q)=1}}\left(\Delta_{f_z}(x_r;q,a_r)-\Delta_{f_z}(a_r;q,a_r)\right)$$

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Proof of Theorem A

- Comparing the two expressions for $f_z(M)$, we find that

$$|\Delta_q| \geq \left(\left| e^{\gamma}\omega\left(\frac{\log U}{\log z}\right) - 1 \right| + O\Big(\frac{1}{\log z}\Big) \right) UV \left(\frac{1}{VP}(F(VP, z)) \right).$$

A key property of the Buchstab function: $\omega(u) - e^{-\gamma}$ has *at most* two zeros in every interval $[u, u+1]$! By restricting the range of U wrt z suitably, we have a constant $C_B > 0$ such that

$$|\Delta_q| \geq \frac{1}{4}\left(C_B + O\Big(\frac{1}{\log z}\Big) \right) UV \left(\frac{1}{VP}(F(VP, z)) \right).$$

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- Comparing the two expressions for $f_z(M)$, we find that

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Versions of Theorem A: f supported on primes

Theorem 2 (Savalia, Vatwani, 2022+)

Fix $A > 1$. Assume that f satisfies $f(n) \ll \Lambda(n)$, hypotheses **H2-H4** and the bound

$$\frac{1}{x}F(x,z) \gg \frac{1}{x^{\frac{1}{2}-\epsilon}},$$

for some $0 < \epsilon < 1/2$. We have the following bounds.

Versions of Theorem A: f supported on primes

Theorem 2 (contd)

(i) Let $c_0 \geq 2$. There exist arbitrarily large values of z, and values of a and x satisfying $z \log z \ll \log x \leq z^{c_0}$, for which

$$\sum_{\substack{q<x/(\log x)^A \\ (q,a)=1}} \left| \sum_{\substack{n\leq x \\ n\equiv a \pmod q}} f_z(n) - \frac{1}{\phi(q)} \sum_{\substack{n\leq x \\ (n,q)=1}} f_z(n) \right| \gg_A \frac{\log z}{\log_2 x} \sum_{n\leq x} f_z(n).$$

(ii) If the summatory function of f satisfies

$$\sum_{n\leq x} f(n) \gg \frac{x}{(\log x)^{C_f}},$$

for some absolute constant C_f, then there are arbitrarily large values of a and x for which we have

$$\sum_{\substack{q\leq \frac{x}{(\log x)^A} \\ (q,a)=1}} \left| \sum_{\substack{n\leq x \\ n\equiv a \pmod q}} f(n) - \frac{1}{\phi(q)} \sum_{\substack{n\leq x \\ (n,q)=1}} f(n) \right| \gg_A \sum_{n\leq x} f(n).$$

Versions of Theorem A: Short intervals

Theorem 3 (Savalia, Vatwani, 2022+)

Fix $A > 1$, $c_0 \geq 2$. Assume that f satisfies hypotheses **H1**, **H3**, **H5** and suitable short interval versions of **H2** and **H4**. Let z be sufficiently large. For any $B > 1$ we have the following bounds.
There exist arbitrarily large values of a, x satisfying

$$z^{1+1/B} < \log x \leq \frac{z^{c_0}}{4},$$

and sufficiently small length of interval $h(x)$ for which

$$\sum_{\substack{q \leq \frac{h}{(\log h)^A} \\ (q,a)=1}} \left| \sum_{\substack{x<n\leq x+h \\ n\equiv a \pmod{q}}} f_z(n) - \frac{1}{\phi(q)} \sum_{\substack{x<n\leq x+h \\ (n,q)=1}} f_z(n) \right|$$

$$\gg_A \log\left(\frac{z^{c_0}}{\log x}\right) \frac{\log\log x}{\log x} \sum_{x<n\leq x+h} f_z(n).$$

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Thank you for your attention!