Cohomology of Quasiperiodic Tilings

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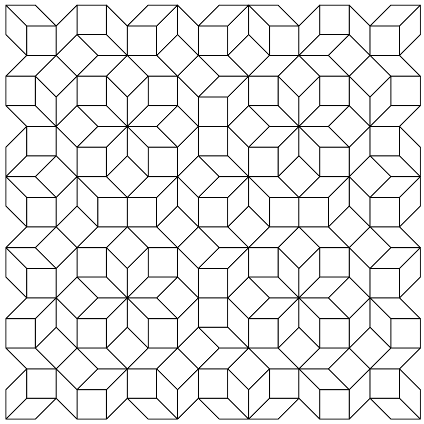
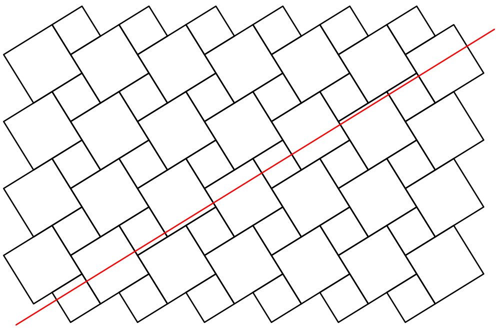
Important Properties of Tilings



Important properties and concepts:

- finite number of local patterns up to translation (finite local comlexity)
- repetitive
- well-defined patch frequencies
- translation module
- local isomorphism (LI classes)
- mutual local derivability

Quasiperiodic Tilings



Irrational sections through a periodic *klotz tiling*.

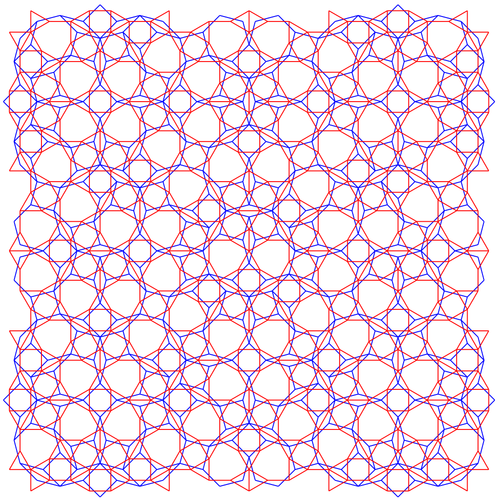
Such tiligs are often called *canonical projection tilings*.

Every vertex, tile, etc, has its *acceptance domain*.

Translation module is projection of higher-dimensional lattice on physical space.

Cut positions touching boundaries of acceptance domains are called *singular*.

Mutual Local Derivability



Two tilings are MLD, if one can be constructed in a local way from the other, and vice versa.

Acceptance domains of one tiling must be constructible by finite unions and intersections of acceptance domains of the other.

MLD induces a bijection between LI classes.

The hull of a tiling

Consider a tiling $\mathcal{T}$ in $\mathbb{R}^d$ of finite local complexity.

We introduce a metric on the set of translates of $\mathcal{T}$:

Two tilings have distance $< \epsilon$, if they agree in a ball of radius $1/\epsilon$ around the origin, up to a translation $< \epsilon$.

The hull $\Omega_\mathcal{T}$ is the closure of $\{\mathcal{T} - x | x \in \mathbb{R}^d\}$ with respect to this metric.

$\Omega_\mathcal{T}$ is a compact metric space, on which $\mathbb{R}^d$ acts naturally by translation.

If $\mathcal{T}$ is repetitive, every orbit is dense in $\Omega_\mathcal{T}$.
$\Omega_\mathcal{T}$ then consists of the LI class of $\mathcal{T}$.

The topology of $\Omega_\mathcal{T}$ is generated by cylinder sets $C(P, x, \epsilon)$, consisting of tilings $\mathcal{T}'$ such that $\mathcal{T}' + y$ has the finite pattern P at x for $y \in B_\epsilon(0)$.

For projection tilings, there is a unique translation invariant measure on $\Omega_\mathcal{T}$.

The measure of cylinder sets is proportional to the frequency of the pattern P (and to the measure of B_ϵ).

Cohomology of the Hull

The cohomology of the hull can be defined and computed in several equivalent ways.

One option is to define it as Cech cohomology, which is what we shall do.

There are explicit formulae for canonical projection tilings of codimension $d^\perp \leq 3$ (Forrest, Hunton, and Kellendonk).

The cohomology groups of canonical projection tilings are free.

Their dimensions are computed from the set of singular cut positions, and how the translation lattice acts on it.

We consider in the following the case $d = d^\perp = 2$, with quadratic embedding of physical space.

Let Γ be the lattice projected to $E^\perp$.

For a canonical projection tiling, the singular set in $E^\perp$ is the union of L_1 Γ-orbits of lines H_α.

The cohomology is finitely generated iff the number L_0 of Γ-orbits of singular points is finite.

Let Γ_α be the stabilizer of H_α in Γ, and L_0^α the number of Γ_α-orbits of singular points on H_α.

Then we have $H^k(\Omega) \cong \mathbb{Z}^{D_k}$, with:

$$D_2 = 3 + L_1 + e - r$$
$$D_1 = 4 + L_1 - r$$
$$D_0 = 1$$

with $e = -L_0 + \sum_\alpha L_0^\alpha$ the Euler characteristic $D_2 - D_1 + D_0$, and r the rank of the span of all $\Lambda_2 \Gamma_\alpha$.

Approximating the Hull by CW-spaces

We define a sequence of cellular CW-spaces Ω_n which approximate Ω.

The d-cells of Ω_0 are the interiors of the tiles.

The lower dimensional cells are the interiors of the tile boundaries; two tile boundaries are identified if they occur as the common boundary of the two tiles somewhere in the tiling.

For Ω_n we proceed as for Ω_0, except that we first color the tiles according to their n^{th} corona.

There is a continuous cellular mapping $h : \Omega_n \to \Omega_{n-1}$, which simply forgets part of the coloring.

We define Ω_∞ as the inverse limit $\varprojlim \Omega_n$, consisting of all sequences $\{x_k\}_{k=0}^\infty$, with $x_k \in \Omega_k$ and $h(x_k) = x_{k-1}$.

The continuous mapping h induces a homomorphism $h_* : H^*(\Omega_n) \to H^*(\Omega_{n+1})$; the cohomology of Ω_∞ is the direct limit

$$H^*(\Omega_\infty) \cong \lim H^*(\Omega_n)$$

It is quite obvious from the construction that Ω and Ω_∞ are homeomorphic, and hence have isomorphic Cech cohomology.

This can be shown also directly.

For each n, Ω has a finite open covering C_n with cylinder sets, whose patterns are the tiles colored according to the n^{th} corona.

From C_n we can construct a finite open covering C'_n of Ω_n, whose nerve is isomorphic to the nerve of C_n.

The seqence of coverings C_n can be chosen such that eventually every open set in Ω is contained in some C_n. Hence:

$$H^*(\Omega) \cong H^*(\Omega_\infty) \cong \lim H^*(\Omega_n)$$

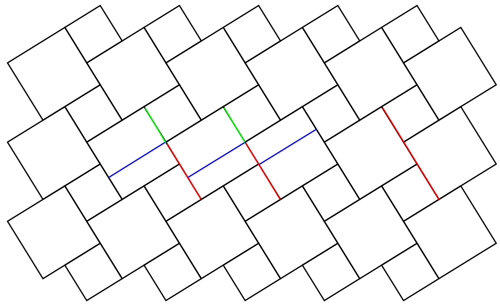
For the cellular spaces Ω_n, the Cech cohomology agrees with the cellular cohomology, and is easily computable.

Our procedure is similar to that of Anderson and Putnam (Ergod. Th. & Dynam. Sys. 18, 509 (1998)), who use a single CW-space Ω', and a mapping $\Omega' \to \Omega'$ induced by inflation. This is equivalent to the refinements according to the n^{th} corona.

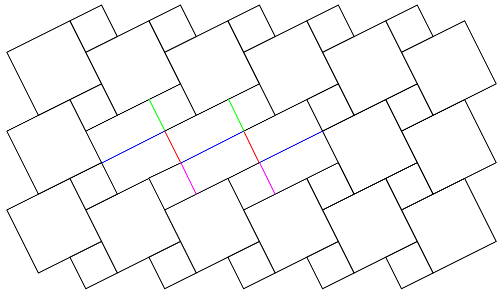
Similar constructions have also been used by Bellissard, Benedetti and Gambaudo (math.DS/0109062).

Application to canonical projection tilings

We first consider the Fibonacci case:



Periodic approximant:



Next we consider the projection from a 3D cubic lattice to a generic one-dimensional line.

Here, the stabilizer Γ_α of each singular line H_α has rank 1.

Hence, there is a rank 2 lattice of singular lines parallel to H_β, and so there are infinitely many Γ_α-orbits of intersections of these with H_α. L_0 is thus infinite, and the cohomology not finitely generated.

How can one see this in our approach?

After some coloring steps, the singular set in 3D consists of continuous, 2D strips of finite width, one for each singular line.

When coloring further, some of these strips will collide, and the cohomology of Ω_n changes.

Since there are infinitely many Γ_α-orbits of singular strips, there will always be more collisions of singular strips, and the cohomology never saturates.

Ammann-Beenker tiling

Also here, the klotz tile boundaries to be extended by the coloring project on singular lines in $E^\perp$.

Let F be the full singular set projecting on a singular line H_α. It is stabilized by a sublattice Γ_α of rank 2.

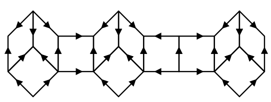
With every klotz tile boundary there is an entire Γ_α-orbit of such boundaries.

If there are finitely many orbits, then after finitely many refinements the topology of the singular set F is that of a thickened 2-plane.

Further refinements won't change it. It is sufficient to consider the n^{th} corona for some fixed, computable n.

There won't be further collisions with other singular 2-planes.

Which corona is required? We have to couple flipping hexagons:



Closing the gaps in the singular set F is also required to make maching rules nearest neighbor ones!

Relation to Matching Rules

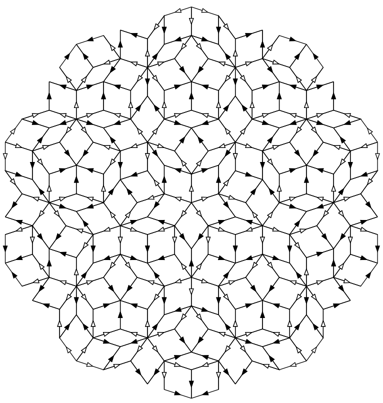
Suppose a canonical projection tiling $\mathcal{T}$ admits local matching rules.

If a decoration is given, which makes the matching rules nearest neighbor local, the CW-space of the decorated (but not further colored) tiles has cohomology isomorphic to that of the hull $\Omega_{\mathcal{T}}$.

Conversely, if the cohomology of the hull $\Omega_{\mathcal{T}}$ is known, this gives a criterion on the decoration being sufficient to make the matching rules nearest neighbor local.

Tiling	H^0	H^1	H^2	e
Penrose	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\mathbb{Z}^8$	4
Tübingen Triangle Tiling	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\mathbb{Z}^{24}$	20
Ammann-Beenker undecorated	$\mathbb{Z}^1$	$\mathbb{Z}^5$	$\mathbb{Z}^9$	5
Ammann-Beenker decorated	$\mathbb{Z}^1$	$\mathbb{Z}^8$	$\mathbb{Z}^{23}$	16
Socolar undecorated	$\mathbb{Z}^1$	$\mathbb{Z}^7$	$\mathbb{Z}^{28}$	22
Socolar decorated	$\mathbb{Z}^1$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{59}$	48

Example: Penrose tiling



Here, there are 4 classes of 0-cells.

The fifth cohomology class counts the number of white arrows in 5 of the 10 directions.

For the Penrose and Ammann-Beenker tiling, H^1 contains more than the reciprocal lattice. It distinguishes inequivalent colorings.

For the Fibonacci chain, H^1 contains just the reciprocal lattice.

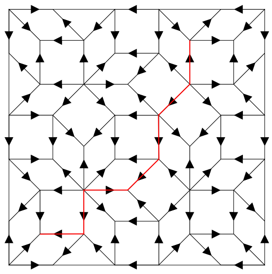
Towards an Interpretation

The advantage of our approach is that the homology and cohomology of Ω_n is given in terms of concrete chains and co-chains of colored tiles.

Consider a continuous sequence of edges in the tiling, representing a 1-chain.

This 1-chain is closed if the two end points are equivalent 0-cells. The homology classes of 1-chains thus include the translation module of the tiling, and the cohomology classes its dual, the reciprocal lattice.

Example: Ammann-Beenker tiling



All 0-cells are equivalent; there is a fifth cohomology class, taking the difference of the number of arrows with and against the orientation of each 1-cell.

Conclusions

The cohomology of the hull is computable from the *local* tile patterns, and how they are arranged.

For canonical projection tilings, tile patterns up to a fixed size are sufficient.

Cohomology classes are explicitly given by representative co-chains of (colored) tiles.

This should help arriving at a more intuitive understanding of these topological invariants.