k-block versus 1-block parallel addition in non-standard numeration systems

Christiane Frougny*, Pavel Heller**,
Edita Pelantová**, Milena Svobodová**

* LIAFA, CNRS UMR 7089 & Université Paris 7 & Université Paris 8, Paris, France
** Dept. of Mathematics, FNSPE, Czech Technical University, Prague, Czech Republic

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Preliminaries

Positional numeration system

Base β and digit set $\mathcal{A}$, where

- $\beta \in \mathbb{C}$, $|\beta| > 1$, algebraic number
- $\mathcal{A} \subset \mathbb{Z}$, finite set of contiguous integers containing 0

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Finite representation of a number

$$z = \sum_{k=-m}^{n} a_k \beta^k \,, \qquad z = a_n \cdots a_0 \bullet a_{-1} \cdots a_{-m}$$

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$$\mathrm{Fin}_{\mathcal{A}}(\beta) = \left\{ \sum_{j \in I} x_j \beta^j \ : \ I \subset \mathbb{Z}, \ \ I \ \text{finite}, \ \ x_j \in \mathcal{A} \right\}$$

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Parallel addition – example

Impossible with $\beta = 10$, $\mathcal{A} = \{0, \ldots, 9\}$:

- $99(9)^n97 + 2 = \;\; 99(9)^n99$
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- $p = 2$

Parallel addition – known results

Theorem (C. Frougny, E. Pelantová, M. Svobodová, 2011)

Let β be an algebraic number such that $|\beta| > 1$ and all its conjugates in modulus differ from 1. Then there exists an alphabet $\mathcal{A} \subset \mathbb{Z}$ such that addition on $\mathrm{Fin}_{\mathcal{A}}(\beta)$ can be performed in parallel.

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Theorem (CF, EP, MS, 2013)

Let β, $|\beta| > 1$, be an algebraic integer with minimal polynomial f. Let $\mathcal{A}$ be an alphabet of contiguous integers containing 0 and 1. If addition in $\mathrm{Fin}_{\mathcal{A}}(\beta)$ is computable in parallel, then $\#\mathcal{A} \geq |f(1)|$. If moreover β is a positive real number, then $\#\mathcal{A} \geq |f(1)| + 2$.

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- β tribonacci base, $\quad f_\beta(X) = X^3 - X^2 - X - 1,$
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- β tribonacci base, $\quad f_\beta(X) = X^3 - X^2 - X - 1$, $\#\mathcal{A} \geq |f_\tau(1)| + 2 = 4$
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Large increase in cardinality of alphabet may be necessary for parallelism.
Potential solution: k-block.

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$$x = x_n \cdots \underbrace{x_{jk+k-1} \cdots x_{jk}}_{X_j} x_{jk-1} \cdots x_k \underbrace{x_{k-1} \cdots x_0}_{X_0} \bullet x_{-1} \cdots x_{-m}$$

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$y \in \mathrm{Fin}_{\mathcal{A}}(\beta)$	$\ldots Y_{j+t} \ldots Y_{j+1} Y_j Y_{j-1} \ldots Y_{j-s} \ldots$	$Y_j \in (\mathcal{A})^k$

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$W_j = X_j + Y_j$	$\dots \underbrace{W_{j+t} \dots W_{j+1} W_j W_{j-1} \dots W_{j-s}} \dots$	$W_j \in (\mathcal{A} + \mathcal{A})^k$

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$W_j = X_j + Y_j$	$\dots \underbrace{W_{j+t} \dots W_{j+1} W_j W_{j-1} \dots W_{j-s}} \dots$	$W_j \in (\mathcal{A}+\mathcal{A})^k$
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$W_j = X_j + Y_j$ $Z_j = \phi(W_{j+t} \ldots W_{j-s})$	$\ldots \underbrace{W_{j+t} \ldots W_{j+1} W_j W_{j-1} \ldots W_{j-s}} \ldots$ $\ldots Z_{j+t} \ldots Z_{j+1} Z_j Z_{j-1} \ldots Z_{j-s} \ldots$	$W_j \in (\mathcal{A} + \mathcal{A})^k$ $Z_j \in (\mathcal{A})^k$

Compare with 1-block:

$x \in \mathrm{Fin}_{\mathcal{A}}(\beta)$ $y \in \mathrm{Fin}_{\mathcal{A}}(\beta)$	$\ldots x_{j+t} \ldots x_{j+1} x_j x_{j-1} \ldots x_{j-s} \ldots$ $\ldots y_{j+t} \ldots y_{j+1} y_j y_{j-1} \ldots y_{j-s} \ldots$	$x_j \in \mathcal{A}$ $y_j \in \mathcal{A}$
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Proposition

Let $\beta \in \mathbb{C}$, $|\beta| > 1$ be an algebraic integer with conjugate γ of modulus $|\gamma| = 1$ and let $\mathcal{A} \subset \mathbb{Z}$ be a finite alphabet. Then no *k*-block *p*-local function can perform parallel addition on alphabet $\mathcal{A}$.

Block parallel addition

Theorem (CF, PH, EP, MS, 2013)

Given a base β and an alphabet $\mathcal{B}$. Let us suppose that there exist positive integers ℓ and r such that for any $x = x_n \ldots x_0 \bullet$ and $y = y_n \ldots y_0 \bullet$ from $\mathrm{fin}_{\mathcal{B}}(\beta)$ the sum $x + y$ has a representation in the form

$$z = x + y = z_{n+\ell} \ldots z_0 \bullet z_{-1} \ldots z_{-r} \, .$$

Then there exists k-block 3-local function performing parallel addition in the alphabet $\mathcal{A} = \mathcal{B} + \mathcal{B}$, where $k = 2(\ell + r)$.

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Take $\beta > 1$, tribonacci base, i.e. root of $x^3 = x^2 + x + 1$.

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Take $\beta > 1$, tribonacci base, i.e. root of $x^3 = x^2 + x + 1$. With the alphabet $\mathcal{B} = \{0, 1\}$, addition in $\mathrm{Fin}_{\mathcal{B}}(\beta)$ is possible with $\ell = 2$ and $r = 5$ (Bernat, 2007).

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The condition might be generally difficult to check, is satisfied in some standard cases.

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Take $\beta > 1$, tribonacci base, i.e. root of $x^3 = x^2 + x + 1$. With the alphabet $\mathcal{B} = \{0, 1\}$, addition in $\mathrm{Fin}_{\mathcal{B}}(\beta)$ is possible with $\ell = 2$ and $r = 5$ (Bernat, 2007). Hence 14-block parallel addition is possible with the alphabet $\mathcal{A} = \{0, 1, 2\}$.

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Proposition (Frougny, Solomyak, 1992)

- A number $\beta > 1$ has Property (PF) if $d_\beta(1) = \bullet t_1 t_2 \cdots t_m$ and $t_1 \geq t_2 \geq \cdots \geq t_m \geq 1$.
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Corollary

Let $d_\beta(1) = t_1 t_2 \ldots t_m$ with $t_1 \ge t_2 \ge t_2 \ge \ldots \ge t_m \ge t \ge 1$ be the Rényi expansion of 1. Then there exists $M \in \mathbb{N}$ such that parallel addition by a k-block local function is possible on the alphabet $\{0, 1, \ldots, M\}$ and $t_1 + t_m \le M \le 2t_1$.

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Block parallel addition – d-bonacci base

Let $d \in \mathbb{N}$, $d \geq 2$. Choose $\beta > 1$ as the real root of $X^d = X^{d-1} + X^{d-2} + \cdots + X + 1$.

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- 1-block parallel addition requires $\#\mathcal{A} \geq |f(1)| + 2 = d + 1$
- $d_\beta(1) = \bullet(1)^d$ and $\lfloor \beta \rfloor = 1$, so k-block parallel addition is possible on the alphabet $\mathcal{A} = \{0, 1, 2\}$. It cannot be further reduced

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- In some cases, estimates on cardinality of alphabet allowing parallel addition k is available
- Minimal size of k is known only in isolated cases (e.g. Penney system)
- Little results on locality of the function, p, are known
- Mutual dependence of the three parameters is yet to be investigated

Thank you for attention