Variational Inequalities and Economic Equilibrium

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Variational inequality representations are set up for a general Walrasian model of consumption and production with trading in a market. The variational inequalities are of functional rather than geometric type and therefore are able to accommodate a wider range of utility functions than has been covered satisfactorily in the past. They incorporate Lagrange multipliers for budget constraints, which are shown to lead to an enhanced equilibrium framework with features of collective optimization. Existence of such an enhanced equilibrium is confirmed through a new result about solutions to nonmonotone variational inequalities over bounded domains. Truncation arguments with specific estimates, based on the data in the economic model, are devised to transform the unbounded variational inequality that naturally comes up into a bounded one having the same solutions.

Key words: Walrasian economic equilibrium; functional variational inequalities; equilibrium computations; equilibrium constraints; complementarity problems

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1. Introduction Concepts of “equilibrium” have long been connected with maximization or minimization. In economic and social situations, game theory has provided formulations in which different entities, or agents, with possibly conflicting interests seek to optimize in circumstances where the actions of any one of them may have consequences for the others. The notion of Nash equilibrium has that form, for instance, as do various models of traffic equilibrium. More complicated versions of equilibrium can to be brought into play for applications where the determination of market prices is crucial. Such applications come up now in areas like electrical power deregulation but also underlie classical economic equilibrium as originally described by Walras.

From a mathematical perspective, it is tempting to think that the many advances in optimization technology in recent decades ought to have major implications for equilibrium problems, not just in analysis but also in computation. Bridges are only starting to be built, however. Within the economics community, studies of equilibrium have centered mainly on existence theorems with increasingly subtle features. Beyond theoretical schemes of *tâtonnment* (see [11] for a later contribution) and the early efforts of Scarf [53, 54] involving fixed-point algorithms (cf. also Todd [55]), relatively little attention has been paid to numerical approaches, not to speak of practical approaches that aim to take advantage of what has been learned in optimization. In the survey of Judd, Kubbler and Schmedders [33], for instance, the computational picture is that of reducing everything to solving large systems of nonlinear equations; see also [32].

In the optimization community, on the other hand, interest in equilibrium problems has definitely taken hold, at least in certain settings which include game equilibria. In some cases linearization methods have been tried [1] and also homotopy methods; cf. [12], [19]. Variational inequality formulations, including complementarity formulations in particular, have been brought in; cf. [1], [15], [36], [38], [39], [40], [29], [23], [22], and the book of Facchinei and Pang [21]. Results on determining a Nash equilibrium have been obtained in [45] and [46]. Meanwhile, work in other quarters has begun on the design and complexity analysis of algorithms for determining equilibrium in pure exchange economies in some special formats not covered just by complementarity; cf. [30], [14], [57]. Just recently, interior-point methods have been put to use [20].

Our intention here is to help this work along by laying out, as a fundamental test case, a fairly general version of Walrasian equilibrium in a format conducive to optimization developments. We want to contribute insights coming from optimization theory, including duality, which could be valuable in making further progress. Part of our goal also is to illuminate the complications that must be surmounted for success in coping broadly with economic equilibrium, inasmuch as most of the work cited above has, for technical reasons, been limited in the scope of the models that are covered. For instance, no producers, only consumers have usually been admitted, and feasible consumption sets have simply been taken to be nonnegative orthants. Utility functions have mostly been assumed to be continuously differentiable even along the boundaries of those orthants, or have been set up so that the boundaries never come into play.

We channel our efforts into showing how classical assumptions, along with the introduction of Lagrange multipliers for translating between prices and consumer utility, allow equilibrium to be represented by a *variational inequality* problem, moreover in a manner which allows treatment of a satisfactory range of utility functions for the economic agents. We show in fact that variational inequalities of "functional" type, generalizing the more common "geometric" type, are needed for this purpose. Variational inequalities of functional type have so far received scant notice among optimizers, although they have an equally long history going back to the 1960's, cf. [35], [5], [6], and for the sake of applications in mechanics and engineering have further been extended to "hemivariational inequalities," which revolve around Clarke generalized subgradients of nonconvex functions, cf. [42], [43], [44]. By exhibiting the versatility of the functional variational inequality format, we hope to arouse interest in solving problems in that format, even beyond the ones described here. Another of our aims is to stimulate the analysis of equilibrium models by way of auxiliary results available for variational inequalities, such as the theory of solution perturbations in [18].

A significant difficulty for our economic application, both for the existence of an equilibrium and the possibilities for computing it, is that the variational inequality we arrive at is *not monotone*. The established literature offers no existence theorem for solutions to nonmonotone variational inequalities of functional type. A result which potentially might be used has recently been formulated in [21, Exercise 2.9.11], but our model is excluded by its assumptions. Without very much difficulty we are able to develop an existence theorem under a simple boundedness condition, but that condition isn't satisfied directly by our variational inequality either, because of inherent unboundedness which arises from the introduction of Lagrange multipliers. However, the possibility arises of a truncation to achieve the boundedness. The struggle then is to demonstrate that, under the economic assumptions we impose, a truncation can be introduced so as to arrive at a bounded variational inequality which has the same solutions as the original one. In our contribution to establishing the existence of economic equilibrium, this is where the really hard work comes in.

After formulating in Section 2 the particular Walrasian model we adopt for economic equilibrium (Definition 1), we lay out in Section 3 the general facts we will need about variational inequalities, including the new existence result for the functional case (Theorem 1). Although our proof of that result comes down in the end to invoking a standard fixed point theorem, we present it in a format suggestive of the forward-backward interations investigated in [13] for monotone variational inequalities of geometric type, even though monotonicity now is lacking.

In Section 4 we get down to the details of the economic model, providing the technical assumptions in full and explaining some of their consequences. In Section 5 we state our results about economic boundedness (Theorem 2) and the existence of a Walrasian economic equilibrium (Theorem 3), comparing them to what has already been known. As a matter of fact, we achieve an equilibrium under assumptions which in some respects are distinctly weaker than those previously in the economic literature (although focusing on concave utility functions to describe preferences). A complementary result about the existence of a kind of ε-equilibrium is presented in Section 5 as well (Theorem 4).

A classical "strong survivability" assumption is shown in Section 6 to guarantee that Lagrange multipliers are available for the so-called budget constraints in the model (Theorem 5). Such multipliers serve to convert utility values for the consumers into market costs and lead to what we call an enhanced economic equilibrium. We demonstrate that an enhanced equilibrium in this sense can be represented by a functional variational inequality (Theorem 6) and moreover has an interesting interpretation involving the maximization of a "collective" utility function formed by assigning weights to the different consumers (Theorem 7).

Section 7 is devoted to the effort of obtaining, from the economic assumptions, estimates of bounds on the elements that enter into an enhanced equilibrium. These bounds support a truncation of the variational inequality to which our general solution criterion can be applied (Theorems 8, 9 and 10). The specific bounds we are able to give for the Lagrange multipliers depend on the strong survivabiity assumption.

Some of the ideas in this paper appeared in our earlier work [31], but with significant differences. Only a pure exchange equilibrium, without producer agents, was treated there. Equilibrium was represented by a geometric variational inequality, but at the expense of seriously limiting the utility functions that could be admitted. Although truncations likewise entered the picture, they were tied to penalties for violations of the budget constraints. A sequence of looser and looser truncations was employed to establish the existence of a "virtual equilibrium" characterized by properties weaker than those in a classical Walrasian equilibrium but approximated arbitrarily closely by them. A concept of virtual equilibrium could similarly be developed in the present context, but we leave that project and its ramifications for another time.

While we hope that our efforts here may eventually lead to the further development of numerical techniques for determining an economic equilibrium, it must be emphasized that the nonmonotonicity of our variational inequality presents a challenge. The existing methodology, such as in [3], [4], [13], [21], [27], [28], relies on monotonicity. It may be envisioned nontheless that, sooner or later, approaches may be found which, in parallel to the use of convex subproblems in nonlinear programming, solve nonmonotone variational inequalities through a sequence of monotone variational inequality subproblems.

2. Equilibrium Model In the classical circumstances we address, there are agents called *consumers*, indexed by $i = 1, \dots, I$, along with agents called *producers*, indexed by $j = 1, \dots, J$. Both deal with "goods," indexed by $k = 1, \dots, K$. Vectors in $\mathbb{R}^K$ having components that stand for quantities of these goods will be involved in both consumption and production. For each consumer i there is a *consumption* set $X_i \subset \mathbb{R}^K$ (also sometimes called a *survival* set), whereas for each producer j there is a *production* set $Y_j \subset \mathbb{R}^K$. Consumer i will choose a consumption vector $x_i \in X_i$, and producer j will choose a production vector $y_j \in Y_j$. Production vectors y_j may have some components negative, for goods that are inputs, and others positive, for goods that are outputs.

Consumption vectors x_i are rated by agent i according their *utility*, which is described by a function u_i on X_i; the higher the utility value $u_i(x_i)$, the better. Each agent i also has an initial *endowment*, a goods vector e_i. Finally, the consumers share in the results of production:

$$\begin{array}{l} \text{consumer } i \text{ gets from } y_j \text{ the fixed fraction } \theta_{ij} y_j, \\ \text{where } \theta_{ij} \geq 0 \text{ and } \sum_{i=1}^{I} \theta_{ij} = 1 \text{ for } j = 1, \dots, J, \end{array} \tag{2.1}$$

The endowment e_i of consumer i is thereby shifted to $e_i + \sum_{j=1}^{J} \theta_{ij} y_j$ as the net vector of goods available to consumer i. It should be noted, however, that negative components of y_j may lead in this way to supply obligations for the consumers: the resulting vector $e_i + \sum_{j=1}^{J} \theta_{ij} y_j$ might itself have some negative components.

Obviously the chosen x_i's and y_j's must turn out to be such that the total consumption $\sum_{i=1}^{I} x_i$ doesn't exceed the total (net) production $\sum_{j=1}^{J} y_j$ plus the total endowment $\sum_{i=1}^{I} e_i$. In other words, consumption and production must somehow be coordinated so that the associated *excess demand* vector

$$z = \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} y_j - \sum_{i=1}^{I} e_i \ \in \mathbb{R}^K. \tag{2.2}$$

actually has no components positive.

The coordination is to be achieved by a market in which goods can be traded (both bought and sold) at particular prices. The prices are *not* part of the given data. Instead, they must be determined from interactions of the consumers and producers over availabilities and preferences. This is why finding an equilibrium is much more than just a matter of optimization.

The price per unit of good k is denoted by p_k; the market is thus governed by a price vector $p = (p_1, \dots, p_K) \in \mathbb{R}^K$. The cost of x_i is $p \cdot x_i = p_1 x_{i1} + \cdots p_K x_{iK}$, whereas the (net) profit from y_j is $p \cdot y_j = p_1 y_{j1} + \cdots + p_K y_{jK}$. We speak of profit because of the convention that output components of y_j are positive and input components are negative.

Most equilibrium models in the economics literature require prices to be nonnegative, although the possibility of negative prices (for goods that turn out to be undesirable, like pollution byproducts) is sometimes admitted. For us, in our wish at this level to avoid unduly troublesome complications en route toward a variational inequality representation, the nonnegativity of prices is highly desirable. Consistent with that, we allow so-called *free disposal* of goods: an agent can dispose of undesired quantities without suffering any penalty. Indeed, this property is virtually equivalent to the a priori exclusion of negative prices from the model, so we embrace it from the start for the convenience it affords.

Definition 1 (Walrasian equilibrium). *An equilibrium consists of a price vector* $\bar{p}$*, consumption vectors* $\bar{x}_i$ *for* $i = 1, \dots, I$*, and production vectors* $\bar{y}_j$ *for* $j = 1, \dots, J$*, such that*

(E1) (market nontriviality) $\bar{p} \geq 0$, $\bar{p} \neq 0$,

(E2) (utility optimization) $\bar{x}_i$ *maximizes* $u_i(x_i)$ *over* $x_i \in X_i$ *subject to the budget constraint*

$$\bar{p} \cdot x_i \leq \bar{p} \cdot \Big[e_i + \sum\nolimits_{j=1}^{J} \theta_{ij} \bar{y}_j \Big], \tag{2.3}$$

(E3) (profit optimization) $\bar{y}_j$ *maximizes* $\bar{p} \cdot y_j$ *over* $y_j \in Y_j$,

(E4) (market clearing) *supplies and demands are balanced in the sense that*

$$\bar{z} \leq 0 \text{ and } \bar{p} \cdot \bar{z} = 0 \quad \text{for } \bar{z} = \sum\nolimits_{i=1}^{I} \bar{x}_i - \sum\nolimits_{j=1}^{J} \bar{y}_i - \sum\nolimits_{i=1}^{I} e_i. \tag{2.4}$$

The case in which there are no producers j (hence no production sets Y_j or shares θ_{ij}) is called a *pure exchange* model of equilibrium. The consumers i start with goods vectors e_i but want to trade them for other goods vectors x_i having higher utility. They are constrained in trading by their wealth, coming from the market value of their endowments. The issue then is whether prices exist under which supplies and demands balance when every agent optimizes utility subject to this budget constraint.

The inequalities, instead of equations, in (2.3) and (2.4) go along with our adoption of free disposal of goods. Ultimately we'll arrive at an equation in (2.3) anyway. When no prices are 0, it's automatic in (2.4) that $\bar{z} = 0$. Price positivity can be guaranteed by extra assumptions on the utilities beyond the ones we'll impose in Section 5, but we won't go into that (cf. [24, 25]).

Elementary presentations often take X_i to be the nonnegative orthant $\mathbb{R}^K_+$, but other possibilities have long been admitted in research on the existence of equilibrium in economics. Although we treat X_i and Y_j abstractly here, these sets can be envisioned as specified by constraints which could further be elaborated in an optimization setting. On the other hand, we are already taking a step away from the abstractions of theoretical economics in posing the preferences of consumers in terms of utilities. Most of the advanced literature in that field revolves around "preference relations" which need not have such an expression, or indeed any other numerical representation. This modeling choice on our part is dictated by our goal of opening the door wider to optimization analysis and computation.

In tandem with the price nonnegativity in (E1), the market clearing requirement in (E4) comes out as a linear *complementarity* condition on $\bar{p}$ and the excess demand vector $\bar{z}$. It says that the supply must exactly equal the demand for a good k with price $\bar{p}_k > 0$, but it can exceed the demand for a good k with price $\bar{p}_k = 0$. Thus, in equilibrium, free disposal is possible only for goods having no value in the market.

Of course, even with complementarity and the stipulation that $\bar{p} \neq 0$, there is nothing in the model to pin down the scale of $\bar{p}$. An equilibrium with price vector $\bar{p}$ is also an equilibrium for any positive multiple of $\bar{p}$. Only the ratios of the positive prices to each other truly matter. For economists, this is as it should be, because various forms of "money" may best be viewed as special goods, the values of which ought to be *derived* from an equilibrium model.

The optimality requirements in (E2) and (E3) might, in some situations, be translated into complementarity conditions as well. This would involve passing from specific constraint representations for X_i and Y_j to optimality conditions in terms of Lagrange multipliers. Nonetheless, there are fundamental obstacles to capturing economic equilibrium entirely through complementarity. The indeterminate scaling of the price vector $\bar{p}$ is one source of trouble in achieving a formulation in which the existence of equilibrium is adequately supported, but deeper difficulties come from the broad range of utility functions that must be admitted in order to maintain a strong connection with economic applications.

Although the existence of a "coordinating" price vector $\bar{p}$ is the central idea in an equilibrium, a kind of of economic feasibility, described next, must be present first. Usually, for technical reasons, a certain boundedness property must be on hand as well. The assumptions we impose later will have to address these concerns.

Definition 2 (economic feasibility and boundedness)**.** *In the context of the model for Walrasian equilibrium, the economy is said to be bounded if the set*

$$A = \Big\{ (x_1, \ldots, x_I, y_1, \ldots, y_J) \,\Big|\, x_i \in X_i,\ y_j \in Y_j,\ \sum_{i=1}^{I} x_i - \sum_{j=1}^{J} y_j - \sum_{i=1}^{I} e_i \leq 0 \Big\} \tag{2.5}$$

is bounded in $(\mathbb{R}^K)^I \times (\mathbb{R}^K)^J$*. The economy is said to be feasible if this set* A *is nonempty, or in other words, if* $z \leq 0$ *for the excess demand vector* z *associated with at least some choice of* $x_i \in X_i$ *and* $y_j \in Y_j$*. The economy is strictly feasible if the choice can made so that actually* $z < 0$ *(where the strict vector inequality refers to strict inequality component by component).*

A customary approach to the issue of price scaling is to restrict price vectors p to lie in the *price simplex*

$$P = \Big\{ p = (p_1, \ldots, p_K) \,\Big|\, p_k \geq 0,\ p_1 + \cdots + p_K = 1 \Big\}, \tag{2.6}$$

so that (E1) turns into $\bar{p} \in P$. For any $\bar{p} \in P$, whether or not part of an equilibrium, conditions (E2) and (E3) can be expressed in terms of

$$\bar{x}_i \in X_i(\bar{p}, \bar{y}_1, \ldots, \bar{y}_J) \ \text{ with } \ \bar{y}_j \in Y_j(\bar{p}), \tag{2.7}$$

where

$$\begin{aligned} Y_j(p) &= \operatorname*{argmax}_{y_j \in Y_j} p \cdot y_j, \\ X_i(p, y_1, \ldots, y_J) &= \operatorname*{argmax}_{x_i \in X_i} \Big\{ p \cdot x_i \leq p \cdot \Big[e_i + \textstyle\sum_{j=1}^{J} \theta_{ij} y_j \Big] \Big\}. \end{aligned} \tag{2.8}$$

This leads to the idea of capturing all the conditions for equilibrium, including (E4), simultaneously in terms of a mapping from price vectors p to goods vectors z, namely the *excess demand* mapping Z having

$$Z(p) = \Big\{ z = \sum_{i=1}^{I} x_i - \sum_{j=1}^{j} y_j - \sum_{i=1}^{I} e_i \,\Big|\, y_j \in Y_j(p),\ x_i \in X_i(p, y_1, \ldots, y_J) \Big\}. \tag{2.8}$$

Clearly

$$\bar{p} \in P \text{ yields equilibrium} \quad \Longleftrightarrow \quad \exists\, \bar{z} \in Z(\bar{p}) \text{ such that } \bar{z} \leq 0,\ \bar{p} \cdot \bar{z} = 0. \tag{2.9}$$

Much of the equilibrium existence theory in economics has focused on the criterion in (2.9), or some variant, in which the compactness of the price simplex P has a major role along with assumptions that guarantee the desired behavior of the mapping Z. The feasibility and boundedness properties in Definition 2 are called upon, in particular, and have to be guaranteed by reasonable assumptions on the sets X_i and Y_j.

Can the excess demand mapping Z be the centerpiece of a *computational* scheme for determining an equilibrium on the basis of (2.9)? This has been investigated by Dafermos [15] through a variational inequality reformulation in the case of a *pure exchange* economy for which Z is *single-valued.* Of course, such single-valuedness depends on assumptions like strict concavity of the utility functions and can't readily be extended to models that incorporate production. Properties like differentiability are even harder to pin down for the mapping Z without serious restrictions, not to speak of utilizing them effectively in an algorithm.

For this reason we won't take that route here, preferring rather to handle $\bar{p}$, $\bar{x}_i$ and $\bar{y}_j$ directly. A variational inequality representation can't be achieved with these elements alone, however. Lagrange multipliers for the budget constraints (2.3) have to be brought in. Those additional elements have economic significance and contribute in a positive way to the notion of equilibrium (we speak of obtaining an "enhanced equilibrium"). But they cause trouble by introducing a further, intrinsic source of unboundedness, even when the economic boundedness in Definition 2 can be guaranteed.

3. Variational Inequalities and Their Solvability A variational inequality of *geometric* type concerns a nonempty, closed, convex set $C \subset I\!R^n$ and a mapping $F : C \to I\!R^n$. The associated problem is

$$\text{VI}(C,F) \qquad \text{find } \bar{v} \in C \text{ such that } -F(\bar{v}) \in N_C(\bar{v}),$$

where $N_C(\bar{v})$ denotes the normal cone to C at $\bar{v}$ in the sense of convex analysis:

$$w \in N_C(\bar{v}) \iff w \cdot (v - \bar{v}) \leq 0 \text{ for all } v \in C. \tag{3.1}$$

If C is a cone, $N_C(\bar{v})$ consists of the vectors w in the polar cone

$$C^* = \{\, w \,|\, w \cdot v \leq 0 \,|\, \text{ for all } v \in C \} \tag{3.2}$$

that satisfy $w \cdot \bar{v} = 0$. The best studied case occurs for $C = I\!R^n_+$, with $C^* = I\!R^n_-$. Then $\text{VI}(C,F)$ reduces to a standard *complementarity* problem:

$$\text{VI}(I\!R^n_+, F) \qquad \text{find } \bar{v} \geq 0 \text{ such that } F(\bar{v}) \geq 0 \text{ and } F(\bar{v}) \perp \bar{v}.$$

In a variational inequality of *functional* type, the set $C \subset I\!R^n$ is replaced by a proper convex function f on $I\!R^n$ that is lower semicontinuous. Properness allows f to take on ∞, as long as the convex set

$$\operatorname{dom} f = \{\, v \,|\, f(v) < \infty \}, \tag{3.3}$$

called the effective domain of f, is nonempty. Lower semicontinuity requires the closedness of all the level sets of the form $\{\, v \,|\, f(v) \leq c \}$ for $c \in I\!R$ (but doesn't necessarily entail $\operatorname{dom} f$ being closed). The problem then is

$$\text{VI}(f,F) \qquad \text{find } \bar{v} \in \operatorname{dom} f \text{ such that } -F(\bar{v}) \in \partial f(\bar{v}),$$

where $\partial f(\bar{v})$ denotes the set of subgradients of f at $\bar{v}$:

$$w \in \partial f(\bar{v}) \iff f(v) \geq f(\bar{v}) + w \cdot (v - \bar{v}) \text{ for all } v \in \operatorname{dom} f. \tag{3.4}$$

We could just as well replace $\operatorname{dom} f$ in these conditions by its closure $\operatorname{cl}(\operatorname{dom} f)$, or for that matter by the whole space $I\!R^n$, since $\partial f(\bar{v})$ is empty when $\bar{v} \notin \operatorname{dom} f$.

The connection between the two types of variational inequalities resides in the fact that the subgradients of f are the normals to C when f is the indicator δ_C of C:

$$\partial f(\bar{v}) = N_C(\bar{v}) \text{ when } f(v) = \delta_C(v) = \begin{cases} 0 & \text{if } v \in C, \\ \infty & \text{if } v \notin C. \end{cases} \tag{3.5}$$

Thus, $\text{VI}(f,F) = \text{VI}(C,F)$ when $f = \delta_C$.

The issue of when a variational inequality is guaranteed to have a solution is critical in guiding our way. Variational inequalities that are *monotone*, i.e., with F satisfying

$$[F(v') - F(v)] \cdot [v' - v] \geq 0 \text{ for all } v, v', \tag{3.6}$$

are especially favorable. They have a highly refined existence theory in finite dimensions, cf. Rockafellar and Wets [52, Chapter 12], along with numerous solution techniques akin to decomposition in convex optimization; cf. Chen and Rockafellar [13] and its references. Infinite-dimensional existence results under monotonicity both types, geometric and functional, go back to the earliest days of the subject in connection with applications to partial differential equations; see [35], [5], [6], [7], [8], [9], [10], [34], [48]. The monotone theory even includes extensions to set-valued mappings F.

Much less is known about *nonmonotone* variational inequalities, however, and this poses a difficulty for us because monotonicity isn't readily available for the variational inequalities that can be used to describe economic equilibrium. In finite dimensions there's a well known existence criterion for the *geometric* case which doesn't invoke monotonicity: if C is bounded (in addition to being closed in $I\!R^n$, hence compact) and F is continuous, then $\text{VI}(C,F)$ has a solution $\bar{v}$; cf. the basic text of Kinderlehrer and Stampacchia [34, p. 12].

This criterion might, in principle, be exploited for economic equilibrium through the characterization in (2.9). One could appeal to the fact that having $\bar{z} \leq 0$ is equivalent to having $p \cdot \bar{z} \leq 0$ for all $p \in P$, and also to a property that follows when the budget constraints (2.3) are sure to be tight at equilibrium, namely that $p \cdot z = 0$ whenever $z \in Z(p)$. Provided that the excess demand mapping Z is single-valued, one could proceed from these observations to an equivalent formulation of (2.9) as concerned with finding

$$\bar{p} \in P \text{ such that } Z(\bar{p}) \cdot (p - \bar{p}) \le 0 \text{ for all } p \in P. \tag{3.7}$$

This characterizes $\bar{p}$ as a solution to $\mathrm{VI}(P, Z)$. In situations where Z is also continuous, the result quoted from [34] could be applied to confirm that such a vector $\bar{p}$ exists.

Unfortunately, things aren't quite so simple, because the sets $X_i(p, y_1, \ldots, y_J)$ in (2.7) and the definition of $Z(p)$ might be empty if certain goods have zero price, and then the domain of Z would not be all of P. Dafermos in [15] found a condition for getting around that and nonetheless obtaining existence, but that condition is beyond direct verification, and there would still be the drawbacks in guaranteeing that Z is single-valued and continuous. On the plus side, however, Dafermos was able to identify in her scheme a class of special models where the variational inequality (3.7) would be monotone.

Economic equilibrium can be represented as a geometric variational inequality $\mathrm{VI}(C, F)$ in other ways than (3.7), but with the underlying set C unbounded and therefore having somehow to be truncated. We have done that in [31], but under strong restrictions on the utility functions u_i which would best be avoided. To get away from those restrictions, we need to deal here instead with functional variational inequalities

What can be said about the existence of solutions for a functional variational inequality $\mathrm{VI}(f, F)$ without monotonicity? Nothing is evident in the literature for cases in which $\operatorname{dom} f$ might not be closed, but we'll be able to fill the gap here by establishing a new result (Theorem 1 below) which relies on a fundamental property of the subgradient mapping ∂f. A result formulated in [21, Exercise 2.9.11] would apply to the case where $f = \delta_C + \varphi$ with C a *closed* convex set and φ a convex function on C which is *continuous* even along the boundary of C. But this wouldn't accommodate the complications associated with the utility functions we need to work with, which will be explained in Section 4.

We present the proof of our existence result for $\mathrm{VI}(f, F)$ in the mode of a forward-backward splitting algorithm of the kind analyzed in Chen and Rockafellar [13]. Without the monotonicity of F (and even some "strong monotonicity" besides), we are unable to say anything about the *practicality* of this algorithm as a numerical method and merely employ it as a technical device for invoking a standard fixed point theorem. But there might be situations where more could be made of it, under further assumptions and analysis. Anyway, as a stepping stone towards a possibly constructive derivation of the existence of equilibrium, this form of argument seems better than the typical nonconstructive fixed-point arguments in the economics literature.

The crucial fact is that the set-valued subgradient mapping ∂f is a *maximal monotone* mapping from $I\!R^n$ to $I\!R^n$; cf. [52, 12.17]. A well known property of any maximal monotone mapping T from $I\!R^n$ to $I\!R^n$ (as utilized in particular in the theory of the proximal point algorithm [50]) is that, for any constant $c > 0$, the "resolvant" mapping $(I + cT)^{-1}$ is single-valued and continuous (actually nonexpansive) from $I\!R^n$ onto the *domain* of T, that domain being by definition the set of v such that $T(v) \neq \emptyset$; see [52, 12.15, 12.19].

Proposition 1 (fixed point representation)**.** *In terms of the mapping* $M = (I + c\partial f)^{-1} \circ (I - cF)$ *for any* $c > 0$*, the solutions* $\bar{v}$ *to problem* $\mathrm{VI}(f, F)$ *are the fixed points of* M*:*

$$-F(\bar{v}) \in \partial f(\bar{v}) \iff \bar{v} \in M(\bar{v}). \tag{3.8}$$

Proof. The left side of (3.8) can equally well be written as $(I - cF)(\bar{v}) \in (I + c\partial f)(\bar{v})$. This is the same as $\bar{v} \in M(\bar{v})$ in view of the single-valuedness of $(I + c\partial f)^{-1}$. □

Theorem 1 (solvability of bounded nonmonotone variational inequalities)**.** *In problem* $\mathrm{VI}(f, F)$*, suppose that* $\operatorname{dom} f$ *is bounded and that* F *is defined and continuous not just on* $\operatorname{dom} f$ *but on* $\operatorname{cl}(\operatorname{dom} f)$*. Then at least one solution* $\bar{v}$ *must exist.*

Proof. Under our assumptions, $\operatorname{cl}(\operatorname{dom} f)$ is a nonempty, compact, convex set, and the single-valued mapping $(I - cF)$ is continuous from $\operatorname{cl}(\operatorname{dom} f)$ into $I\!R^n$. On the other hand, the single-valued mapping $(I + c\partial f)^{-1}$ is continuous from $I\!R^n$ onto the set of v for which $\partial f(v) \neq \emptyset$, which is a subset of $\operatorname{dom} f$.

The composed mapping $M = (I + c\partial f)^{-1} \circ (I - cF)$ is therefore continuous and carries $\operatorname{cl}(\operatorname{dom} f)$ into itself. Hence, by the Brouwer fixed-point theorem, it has a fixed point $\bar{v}$, which by Proposition 1 is a solution to problem $\mathrm{VI}(f, F)$. □

Theorem 1 reduces to the result quoted from Kinderlehrer and Stampacchia [34] for geometric variational inequalities $\mathrm{VI}(C, F)$ with compact C when specialized through (3.9).

From the algorithmic perspective, one can imagine trying to locate a fixed point of the mapping M in this framework by iterating with $v_{k+1} = M(v_k)$, starting from some $v_0 \in \operatorname{dom} f$. Such iterations have a "forward-backward" character, because they first take a "forward" step from v_k to $v'_k = [I - cF](v_k) = v_k - cF(v_k)$ (think of $-F$ as giving a direction in which to move and c as the step size), and second take a "backward" step by determining $v_{k+1} = (I + c\partial f)^{-1}(v'_k)$ in effect as a solution to the relation $v_k \in v_{k+1} + c\partial f(v_{k+1})$; this means

$$v_{k+1} \in \operatorname{argmin}_v \Big\{ f(v) + \frac{1}{2c}|v - v_k|^2 \Big\}, \tag{3.9}$$

where $|\cdot|$ denotes the Euclidean norm in $I\!R^n$. Thus, the resolvent $(I + c\partial f)^{-1}$ can readily be handled in a computational environment.

In the case of a geometric variational inequality as in (3.5), the backward step is executed by taking v_{k+1} to be the projection of $v_k - cF(v_k)$ on C. To help understand the picture, note that if F were the gradient mapping ∇g of a smooth function g, for instance, the forward step $v'_k = v_k - cF(v_k)$ would correspond to movement in the direction of steepest descent.

Many variants can be contemplated, for instance with c replaced by a different c_k in each iteration, or even with the identity I replaced by a varying positive-definite, symmetric H_k; cf. [13]. Again, though, whether a good numerical procedure can be put together from this is unclear and not our topic here.

For comparison with some of the methodology for complementarity problems another observation can be made. Recall that any lower semicontinuous, proper, convex function f on $I\!R^n$ is conjugate to another such function f^* under the relations

$$f^*(w) = \sup \Big\{ w \cdot v - f(v) \Big\}, \qquad f(v) = \sup \Big\{ w \cdot v - f^*(w) \Big\}. \tag{3.10}$$

Subgradients are connected to conjugacy through the fact that

$$f(v) + f^*(w) - v \cdot w \begin{cases} \geq 0 & \text{for all } (v, w) \in I\!R^n \times I\!R^n, \\ = 0 & \text{if and only if } w \in \partial f(v), \end{cases} \tag{3.11}$$

cf. [52, 11.5]. This yields a basic characterization of solutions to variational inequalities as solutions to (nonconvex) problems of optimization.

Proposition 2 (variational inequalities as optimization)**.** *In relation to problem* $\mathrm{VI}(f, F)$*, let*

$$\varphi(v) = f(v) + f^*(-F(v)) + v \cdot F(v) \ \text{ for } \ v \in \operatorname{dom} f. \tag{3.12}$$

Then $\varphi \geq 0$*, and in order that* $\bar{v}$ *solve* $\mathrm{VI}(f, F)$*, it is necessary and sufficient that*

$$\bar{v} \in \operatorname*{argmin}_{v \in \operatorname{dom} f} \varphi(v), \quad \text{with} \quad \min_{v \in \operatorname{dom} f} \varphi(v) = 0. \tag{3.13}$$

Proof. This is immediate from (3.11). □

Observe that for a standard complementarity problem, which corresponds to

$$f = \delta_{\boldsymbol{R}^n_+}, \qquad f^* = \delta_{\boldsymbol{R}^n_-}, \tag{3.14}$$

Proposition 2 gives the familiar characterization of solutions in terms of minimizing $v \cdot F(v)$ subject to $v \geq 0$ and $F(v) \geq 0$. Also to note as furnishing a potentially interesting case in (3.12) is the fact that

$$\begin{array}{c} f^*(w) = f(-w) \ \text{ when } \ f(v_1, \ldots, v_n) = -\sum_{i=1}^n \alpha_i \log v_i - \beta \\ \text{with } \ \alpha_i > 0, \ \sum_{i=1}^n \alpha_i = 1, \ \ \beta = \frac{1}{2}\big[1 - \sum_{i=1}^n \alpha_i \log \alpha_i\big], \end{array} \tag{3.15}$$

under the convention that $\log t = -\infty$ when $t \leq 0$. This case has ties to the Cobb-Douglas utility functions often used in economic modeling (see below), as well as to barrier treatments of inequality constraints.

Whether optimization techniques already developed in the complementarity setting can find extensions to functional variational inequalities of some kinds through Proposition 2, remains to be seen. Cases like (3.15) where f is separable might be especially worth investigating, since then f^* is easy to determine and is separable as well; cf. [52, 11.22], [51, Chapter 8]. As another possibly useful tool, the condition $v \cdot F(v) \leq \varepsilon$ that appears in association with complementarity generalizes in Proposition 2 to $f(v) + f^*(-F(v)) + v \cdot F(v) \leq \varepsilon$, which means in convex analysis that $-F(v)$ belongs to $\partial_\varepsilon f(v)$, the set of ε-*subgradients* of f at v; see [47, pp. 219–220].

4. Economic Assumptions Returning now to the economic setting, we work toward the statement of an existence theorem for Walrasian equilibrium that will fit with our variational inequality framework.

An important consideration, leading to a degree of subtlety and novelty in our choice of assumptions, is the need to take a wide range of utility functions u_i into account. Although *linear* utility functions, with

$$u_i(x_i) = \sum\nolimits_{k=1}^{K} c_{ik} x_{ik} \text{ for } x_i = (x_{i1}, \ldots, x_{iK}), \tag{4.1}$$

can be useful for theoretical and computational experiments (cf. [26], [30], [57]), they may be unconvincing as descriptions of an agent's preferences. In general, if we wish maintain a strong bond with applications in economics, we have to be careful not to suppose too much, especially about how u_i behaves on the boundary of X_i.

A common class of utility functions which is very much appreciated in economics consists of those of *Cobb-Douglas type*, where

$$u_i(x_i) = x_{i1}^{\alpha_{i1}} x_{i2}^{\alpha_{i2}} \cdots x_{iK}^{\alpha_{iK}} \text{ for } x_i \in I\!R^K_+ \text{ where } \alpha_{ik} \geq 0, \ \sum\nolimits_{k=1}^{K} \alpha_{ik} = 1. \tag{4.2}$$

Further, there is the class of utility functions having *constant elasticity of substitution* (CES), where the formula is

$$u_i(x_i) = \Big[\sum\nolimits_{k=1}^{K} [c_{ik} x_{ik}]^\alpha \Big]^{1/\alpha} \text{ for } x_i \in I\!R^K_+ \text{ with } \alpha \in (0,1) \text{ or } \alpha \in (-\infty, 0) \tag{4.3}$$

with coefficients $c_{ik} \geq 0$. When $X_i = I\!R^K_+$, both of these classes raise questions about boundary behavior.

Cobb-Douglas utilities, while unambiguously defined and continuous on all of $I\!R^K_+$, are only differentiable on the interior of $I\!R^K_+$. Their gradients $\nabla u_i(x_i)$ get unbounded as x_i tends to the boundary of $I\!R^K_+$, at least in some places. On the other hand, the rule of positive homogeneity holds, i.e., $u_i(\lambda x_i) = \lambda u_i(x_i)$ for all $\lambda > 0$, and this has the consequence that $\nabla u_i(\lambda x_i) = \nabla u_i(x_i)$ for all $\lambda > 0$. Thus, gradient limits along linear approaches to the origin exist but can differ, depending on the direction of approach; there's inherent discontinuity in the behavior of the gradient mapping ∇u_i at the origin.

For CES utilities, likewise positively homogeneous, there are similar characteristics, but also, when $\alpha < 0$, additional trouble over making sense of $[c_{ik} x_{ik}]^\alpha$ if $c_{ik} x_{ik} = 0$. This can largely be handled by taking $u_i(x_i)$ to be $-\infty$ in those cases, but not entirely, since the function's behavior of along linear approaches to the origin dictates having $u_i(0) = 0$. Utility functions with logarithmic terms can appear also. Then $u_i(x_i)$ may tend to $-\infty$ no matter how the boundary of $I\!R^K_+$ is approached from its interior.

It's essential, therefore, to admit various utility functions u_i which, on the boundary of X_i, might lack not only differentiability but also continuity, and which may even take on $-\infty$. At such a level of generality a simple complementarity representation, or even a standard geometric variational inequality representation, is unattainable. The situation is not so bleak as it may seem, though, because all the utility functions mentioned are concave and upper semicontinuous on $I\!R^K_+$ (when properly interpreted). Functions in that category are fully understood from the standpoint of convex analysis and have many features which can be exploited advantageously in optimization.

With this background, we are ready to state the assumptions on which our investigations will proceed. Although weaker assumptions might suffice for some purposes, we forgo aspects of generality that may be more apparent than real. We concentrate on features capable of producing the explicit bounds which will serve in obtaining the existence of an equilibrium by way of a variational inequality, indeed, a functional variational inequality that is open to truncation so that Theorem 1 can successfully be applied.

Basic Assumptions.

(A1) *Each consumption set X_i is a nonempty, closed, convex subset of $I\!R^K_+$.*

(A2) *Each utility function u_i is concave and upper semicontinuous on X_i and finite on the interior of X_i, although it may take on $-\infty$ on the boundary of X_i.*

(A3) *If $x_i \in X_i$ and $x'_i \geq x_i$, then $x'_i \in X_i$ and $u_i(x'_i) \geq u_i(x_i)$; in particular, X_i has nonempty interior. Furthermore, u_i is insatiable in the sense of not attaining a maximum on X_i.*

(A4) *Each production set Y_j is a nonempty, closed, convex subset of $I\!R^K$.*

(A5) *There is at least one price vector $\hat{p} > 0$ such that the set $Y_j(\hat{p}) = \operatorname{argmax}\{\, \hat{p}\cdot y_j \,|\, y_j \in Y_j \,\}$ is nonempty and bounded for every producer j,*

(A6) *For each i there is a consumption vector $\hat{x}_i \in X_i$ that satisfies $\hat{x}_i < e_i + \sum_{j=1}^{J} \theta_{ij} \hat{y}^i_j$ for some choice of production vectors $\hat{y}^i_j \in Y_j$.*

Some remarks on these assumptions are now in order. The closedness and convexity of X_i are standard. We would not really have to place X_i within $I\!R^K_+$, but with an eye toward achieving the boundedness in Definition 2 we would anyway be led to supposing the existence of a lower bound vector x^-_i such that $x_i \geq x^-_i$ for every $x_i \in X_i$ (as seen in classical theory). Then, however, we could make a change of variables from x_i to $x'_i = x_i - x^-_i$ and translate X_i into a set X'_i lying in $I\!R^K_+$, while replacing the endowment e_i by $e'_i = e_i = x^-_i$. Thus, we can just as well suppose from the beginning that $X_i \subset I\!R^K_+$.

We have already given reasons for caution in imposing conditions of continuity or even finiteness on the utility functions, and this is reflected in (A2). The finiteness and concavity of u_i on the interior of X_i ensure at least that u_i is continuous on that open convex set, and that its values at boundary points, whether finite or infinite, are determined by the limits of u_i along line segments from the interior; see [52, 2.35]. It's good to observe, however, that only the *preference mapping* P_i associated with u_i, which assigns to each $x_i \in X_i$ the set

$$P_i(x_i) = \{\, x'_i \in X_i \,|\, u_i(x'_i) > u_i(x_i) \,\}, \tag{4.4}$$

really influences the consumer's optimization. This leads to some flexibility which could well be helpful in practice. Replacing u_i by $\varphi_i \circ u_i$ for an increasing function φ_i has no effect on the maximization in (E2), and it preserves concavity and upper semicontinuity when φ_i is concave (and continuous) on the range of u_i (with $\varphi_i(-\infty)$ interpreted as $-\infty$, if need be); cf. [52, 2.20(b)].

For example, by composing a Cobb-Douglas utility function u_i as in (4.2) with $\varphi(u) = \log u$, one obtains, as an "equivalent" utility for equilibrium purposes, the *separable* concave function

$$\tilde{u}_i(x_i) = \alpha_{i1} \log x_{i1} + \cdots + \alpha_{iK} \log x_{iK} \tag{4.5}$$

(with $\log 0 = -\infty$). When every α_{ik} is positive, the gradients of this equivalent utility unfailingly blow up as the boundary of $I\!R^K_+$ is approached, and so too do the function values, which on the boundary must everywhere be taken as $-\infty$. This is simpler than the behavior of u_i itself.

The insatiability in (A3) follows a tradition in economics and has a definite role in obtaining an equilibrium. The monotonicity in (A3) is not a true restriction, in the light of our decision to allow free disposal of goods. When $x'_i \geq x_i \in X_i$, one can revert from x'_i to x_i by getting rid of the difference, so it's unrealistic to think of x'_i as possibly suffering disadvantages relative to x_i from the perspective of consumer i.

To appreciate (A5), which will be a critical ingredient for us in obtaining the economic boundedness property in Definition 2, recall that an equilibrium already entails in (E3) the existence of a nonzero price vector $\bar{p}$ for which the sets $Y_j(\bar{p})$ for $j = 1, \ldots, J$ are all nonempty. Here we are asking for something only a little stronger, in terms of a price vector $\hat{p}$ that needn't, itself, be part of an equilibrium.

The equivalences in the proposition proved next shed further light on the nature of (A5). In this we make use of the *recession cones* Y^∞_j of the convex sets Y_j, expressed by

$$Y^\infty_j = \left\{\, y^\infty_j \in I\!R^K \;\middle|\; y_j + \tau y^\infty_j \in Y_j \text{ for all } y_j \in Y_j \text{ and } \tau > 0 \,\right\}, \tag{4.6}$$

[47, Sec. 8], [52, Sec. 2G], as well the recession cone Y^∞ associated with the convex set

$$Y = Y_1 + \cdots + Y_J \quad \text{(the \textit{total} production set).} \tag{4.7}$$

The recession cone of any *closed* convex set is a *closed* convex cone, but Y conceivably might not be closed. Guaranteeing the closedness of Y is one of the virtues of (A5), as we now show in particular.

Proposition 3 (equivalent conditions on production)**.** *In the presence of the closedness and convexity of each Y_j as in (A4), assumption (A5) is equivalent to each of the following:*

(A5′) *The set of $p \geq 0$ such that $Y_j(p) \neq \emptyset$ for all j has nonempty interior,*

(A5″) *The set of $p \geq 0$ such that $\sup\{\, p \cdot y_j \,|\, y_j \in Y_j \} < \infty$ for all j has nonempty interior.*

(A5‴) *Y is a closed convex set having $Y^\infty \cap (-Y^\infty) = \{0\}$ and $Y^\infty \cap \mathbb{R}^K_+ = \{0\}$.*

Proof. Let $S_j(p) = \sup\{\, p \cdot y_j \,|\, y_j \in Y_j \}$ for all $p \in \mathbb{R}^K$; this is the support function of the nonempty, closed, convex set Y_j. It has the following general properties, cf. [47, Sec. 13, 23.5.3], [52, 8.24, 8.25]. It's a lower semicontinuous, positively homogeneous, convex function on $\mathbb{R}^K$. Its effective domain $\operatorname{dom} S_j$ is a convex cone (not necessarily closed), having Y_j^∞ as its polar. The subgradient set $\partial S_j(p)$ for any p is $Y_j(p)$. The set of p for which $\partial S_j(p)$ is nonempty is a subset of $\operatorname{dom} S_j$ that includes the relative interior of $\operatorname{dom} S_j$; furthermore, $\partial S_j(p)$ is both nonempty and bounded if and only if p belongs to the interior of $\operatorname{dom} S_j$. In consequence of these well known facts of convex analysis, (A5), (A5′) and (A5″) can all be identified with the property that

$$\text{there exists } \hat{p} \in \Big[\bigcap\nolimits_{j=1}^{J} \operatorname{int}(\operatorname{dom} S_j) \Big] \cap \operatorname{int}(\mathbb{R}^K_+). \tag{4.8}$$

We must show that this property is equivalent in turn to (A5‴).

The set $Y = \sum_{j=1}^J Y_j$ is nonempty and convex by virtue of the nonemptiness and convexity of each Y_j. Let S be the support function of Y (or equivalently that of $\operatorname{cl} Y$), so that $S = \sum_{j=1}^J S_j$, $\operatorname{dom} S = \cap_{j=1}^J \operatorname{dom} S_j$, and furthermore $\operatorname{int}(\operatorname{dom} S) = \cap_{j=1}^J \operatorname{int}(\operatorname{dom} S_j)$ if either side is nonempty.

In these terms, (4.8) means that $\operatorname{dom} S$ has nonempty interior and $\operatorname{dom} S$ can't be separated from $\mathbb{R}^K_+$. If Y is closed, we have Y^∞ as the polar of $\operatorname{dom} S$, and these two conditions on $\operatorname{dom} S$ dualize to the two conditions in (A5‴). Since the nonemptiness of $\cap_{j=1}^J \operatorname{int}(\operatorname{dom} S_j)$ is known to be sufficient for $\sum_{j=1}^J Y_j$ to be closed (e.g. by [47, Thm. 16.5] as applied to the functions S_j, which have $S_j^* = \delta_{Y_j}$), it follows that (4.8) holds if and only if (A5‴) holds. □

Finally, we come to the so-called *strong survivability* assumption (A6), a traditional condition introduced by Arrow and Debreu [2] which can be contrasted to (plain) *survivability*, the alternative assumption in which the strict inequality in (A6) is reduced to weak inequality:

(A6⁻) *For each i there is a consumption vector $\hat{x}_i \in X_i$ that satisfies $\hat{x}_i \leq e_i + \sum_{j=1}^J \theta_{ij} \hat{y}^i_j$ for some choice of production vectors $\hat{y}^i_j \in Y_j$.*

From the angle of optimization, (A6) can be regarded as kind of Slater condition with respect to the fundamental limitations on production and consumption. In a pure exchange equilibrium (without production), survivability means the agents can simply "stay at home," whereas strong survivability, in contrast, is less palatable because it insists on every agent i having, from the beginning, a *positive quantity of every good*, regardless of its utility to that agent.

Nevertheless, strong survivability is a valuable condition for achieving the existence of an equilibrium. Many researchers have tried to get around it, but at the expense of relying on assumptions of an abstract character which would be hard to check from the data, or by accepting a looser concept of what constitutes equilibrium. Our interests lie in the opposite direction, looking toward an "enhanced" concept of equilibrium defined below. We therefore accept strong survivability as a platform for the developments in this paper and try to make the most from it.

5. Basic Economic Results

The fundamental consequences of our assumptions on the economic model will now be laid out.

Proposition 4 (guarantee of economic feasibility)**.** *The strong survivability assumption (A6), together with (A4), ensures economic feasibility in the sense of Definition 2. (In fact, it ensures strict feasibility; (A6⁻) would already be enough for feasibility.)*

Proof. If $\hat{x}_i$ and $\hat{y}^i_j$ satisfy (A6⁻), we have

$$\sum_{i=1}^{I} \hat{x}_i - \sum_{i=1}^{I} \hat{e}_i \le \sum_{i=1}^{I} \Big[\sum_{j=1}^{J} \theta_{ij}\hat{y}_j\Big] = \sum_{j=1}^{J} \Big[\sum_{i=1}^{I} \theta_{ij}\hat{y}_j\Big]. \tag{5.1}$$

Then, by defining $\hat{y}_j = \sum_{i=1}^{I} \theta_{ij}\hat{y}_j$ and observing through (2.1) that $\hat{y} \in Y_j$ by (A4), we see we have $(\ldots, \hat{x}_j, \ldots; \ldots, \hat{y}_j, \ldots)$ belonging to the set A in (2.5), hence feasibility. If $\hat{x}_i$ and $\hat{y}^i_j$ actually satisfy (A6), the inequality in (5.1) is strict and we arrive at strict feasibility. □

The facts in Proposition 4 are well appreciated in economics, but we have supplied the brief argument anyway for completeness. Note that (A6) isn't *necessary* for strict economic feasibility, just *sufficient*.

Theorem 2 (guarantee of economic boundedness)**.** *Under (A1), (A4) and (A5), the economy is certain to be bounded in the sense of Definition 2.*

Proof. The set A in (2.5) is a closed, convex subset of $(\mathbb{R}^K)^I \times (\mathbb{R}^K)^J$. In showing it must be bounded, we can suppose it to be nonempty (as will anyway hold through Proposition 4). Its recession cone then has the formula

$$A^\infty = \Big\{ (x_1^\infty, \ldots, x_I^\infty, y_1^\infty, \ldots, y_J^\infty) \,\Big|\, x_i^\infty \in X_i^\infty,\ y_j^\infty \in Y_j^\infty,\ \sum_{i=1}^{I} x_i^\infty \le \sum_{j=1}^{J} y_j^\infty \Big\}. \tag{5.2}$$

A necessary and sufficient condition for A to be bounded is that A^∞ contains nothing more than the zero vector [47, Theorem 8.4]. Consider therefore any $w = (x_1^\infty, \ldots, x_I^\infty, y_1^\infty, \ldots, y_J^\infty)$ in A^∞. Since $X_i \subset \mathbb{R}^K_+$, we have $X_i^\infty \subset \mathbb{R}^K_+$, hence $x_i^\infty \ge 0$ for $i = 1, \ldots, I$. From the inequality condition in (5.2), it follows that $\sum_{j=1}^{J} y_j^\infty \ge 0$. Let $\hat{p}$ be a price vector as in (A5). Because $y_j + y_j^\infty \in Y_j$ for every $y_j \in Y_j$, we must have $\hat{p} \cdot y_j^\infty < 0$ for any j such that $y_j^\infty \ne 0$, for otherwise the nonemptiness and boundedness of $Y_j(\hat{p})$ in (A5) would be contradicted. On the other hand, we have $\sum_{j=1}^{J} \hat{p} \cdot y_j^\infty \ge 0$ because $\sum_{j=1}^{J} y_j^\infty \ge 0$ and $\hat{p} > 0$. Necessarily, then, $y_j^\infty = 0$ for every j. The inequality in (5.2) then implies $\sum_{i=1}^{I} x_i^\infty \le 0$, where however $x_i^\infty \ge 0$, so we conclude that $x_i^\infty = 0$ for every i as well. □

This result can be compared to the boundedness criterion of Debreu in [17]. There, the sets X_i are required to be "bounded from below," which can harmlessly be reduced to the stipulation in (A1) that $X_i \subset \mathbb{R}^K_+$, in the manner explained earlier. On the other hand, in [17] assumptions aren't placed on the sets Y_j themselves, but only on the total production set Y, the requirement being that

$$Y \text{ is closed and convex with } Y \cap (-Y) = \{0\} \text{ and } Y \cap \mathbb{R}^K_+ = \{0\}. \tag{5.3}$$

This condition on Y can compared to our (A5) through its equivalence with (A5‴) in Proposition 3. There would be no real loss of generality in adding to (A4) the assumption that $0 \in Y_j$ for every producer j (since it's possible always to achieve this by a translation), in which case Y_j^∞ is the greatest closed, convex cone included in Y_j. Our condition (A5‴) clearly emerges then as less demanding than Debreu's. If the production sets Y_j are convex cones, one has $Y_j^\infty = Y_j$ and the two conditions come down to the same thing.

Theorem 3 (existence of Walrasian equilibrium)**.** *Under assumptions (A1)–(A6) an equilibrium in the sense of Definition 1 is sure to exist.*

Theorem 3 will be proved in Section 7 by an argument based "constructively" on Theorem 1 as well as on Theorem 2 and other boundedness properties; see Theorem 9. It's close in spirit to known results in economics going back to Arrow and Debreu [2], [16], [17], and others, but isn't covered by those results because of differences in context and assumptions.

In relying on concave utility functions for expressing preferences, for instance, Theorem 3 is more special than the other existence theorems, but in merely demanding the upper semicontinuity of such functions, without even insisting on them being finite everywhere, it's more general. A relatively recent exposition in [41], which could serve as an up-to-date standard for the existence of Walrasian equilibrium without getting into further, less classical features, is formulated in terms of preference sets which, for us, would be the sets in (4.4). The result in [41] depends on supposing that all such sets are open relative to X_i. But that property isn't guaranteed by our utility assumptions in (A2) and (A3); it would force u_i to be continuous relative to X_i, instead of just upper semicontinuous. Also, the economic boundedness needed in the background of equilibrium is just *assumed* in [41], with a wave toward Debreu's criterion, whereas we furnish through Theorem 2 a milder criterion which is better suited to computational developments.

Our approach further provides a complementary result about the existence of an "approximate" equilibrium in a certain sense.

Definition 3 (ε-equilibrium with budget adjustments)**.** *By an ε-equilibrium for any $\varepsilon > 0$ will be meant a Walrasian equilibrium as in Definition 1, except that the budget constraint in (E2) is changed for each agent i to*

$$\bar{p}\cdot x_i \leq \bar{p}\cdot\Big[\, e_i + \sum\nolimits_{j=1}^{J} \theta_{ij}\bar{y}_j \,\Big] + \pi_i \tag{5.4}$$

through a choice of adjustment values π_i satisfying

$$\sum\nolimits_{i=1}^{I} \pi_i = 0, \qquad \sum\nolimits_{i=1}^{I} |\pi_i| \leq \varepsilon||\bar{p}||_1. \tag{5.5}$$

The budget adjustments π_i in an ε-equilibrium can be viewed in terms an arbitrarily small redistribution of the total wealth in the economy (under the price vector $\bar{p}$) through "credits" (amounts $\pi_i > 0$) and "debits" (amounts $\pi_i < 0$) relative to the market values of the holdings of the various consumers. Normalization of $\bar{p}$ to belong to the simplex P would correspond to scaling $\bar{p}$ so that $||\bar{p}||_1 = 1$. Then in (5.5) the total of the credits must equal the total of the debits, and this common total can't exceed $\varepsilon/2$.

Theorem 4 (existence of ε-equilibrium)**.** *Under (A1)–(A5), but with (A6) relaxed by the combination of the survivability in (A6$^-$) with the direct assumption of strict economic feasibility, an ε-equilibrium in the sense of Definition 3 is sure to exist for every $\varepsilon > 0$.*

The proof of Theorem 4, like that of Theorem 3, will be given in Section 7; see Theorem 10. It should be noted that an ε-equilibrium as defined here differs from the type of approximate equilibrium appearing in the recent computational proposals of [30], [14], [57].

6. Enhanced Equilibrium and its Representation The key to our way of representing a Walrasian equilibrium as a solution to a variational inequality is the introduction of Lagrange multipliers for the budget constraints (2.3).

Definition 4 (enhanced Walrasian equilibrium)**.** *An equilibrium in the sense of Definition 1 will be called an equilibrium enhanced by utility scale factors if it satisfies in place of* (E2) *the following, generally stronger condition for each consumer i:*

(E2$^+$) (enhanced utility optimization) *There exists a scalar $\bar{\lambda}_i > 0$ such that $\bar{x}_i$ maximizes $u_i(x_i) - \bar{\lambda}_i\bar{p}\cdot x_i$ over $x_i \in X_i$ without reference to the budget constraint (2.3), and $\bar{x}_i$ nonetheless satisfies that budget constraint, moreover tightly:*

$$\bar{p}\cdot x_i = \bar{p}\cdot\Big[\, e_i + \sum\nolimits_{j=1}^{J} \theta_{ij}\bar{y}_j \,\Big]. \tag{6.1}$$

Likewise under the substitution in Definition 4, except with π_i added to the right side of (6.1), an ε-equilibrium will be called an *ε-equilibrium enhanced by utility scale factors.*

A coefficient $\bar{\lambda}_i$ as described in (E2$^+$) can appropriately be called a *utility scale factor* for consumer i, because it converts the cost of acquiring x_i in the market, namely $\bar{p}\cdot x_i$, into units of utility of agent i which can be balanced off against the utility value $u_i(x_i)$.

The elementary fact that (E2$^+$) implies (E2) is evident from the observation that the maximization in (E2$^+$) entails having $u_i(x_i) - u_i(\bar{x}_i) \leq \bar{\lambda}_i\bar{p}\cdot[\bar{x}_i - x_i]$ for all $x_i \in X_i$, while on the other hand $\bar{p}\cdot[\bar{x}_i - x_i] \leq 0$ when x_i satisfies (2.3) and $\bar{x}_i$ satisfies (6.1).

Together with $\bar\lambda_i$ being positive, the conditions on $\bar x_i$ and $\bar\lambda_i$ in (E2$^+$) say that $(\bar x_i, \bar\lambda_i)$ is a saddle point of the Lagrangian function

$$\bar L_i(x_i,\lambda_i) = u_i(x_i) + \lambda_i\big[\bar w_i - \bar p\cdot x_i\big] \quad \text{for } x_i\in X_i,\ \lambda_i\in[0,\infty), \tag{6.2}$$

with respect to maximizing in x_i and minimizing in λ_i, where $\bar w_i$ is the *wealth* of consumer i as derived from the price vector $\bar p$ by

$$\bar w_i = \bar p\cdot\Big[\, e_i + \sum\nolimits_{j=1}^J \theta_{ij}\bar y_j\,\Big] = \bar p\cdot e_i + \sum\nolimits_{j=1}^J \theta_{ij}\bar p\cdot\bar y_j. \tag{6.3}$$

The saddle point condition by itself means that

$$\begin{cases} \bar x_i \in \operatorname{argmax}_{x_i\in X_i}\big\{\, u_i(x_i) + \bar\lambda_i\big[\bar w_i - \bar p\cdot x_i\big]\,\big\}, \\ \bar\lambda_i\geq 0,\quad \bar w_i - \bar p\cdot x_i \geq 0,\quad \bar\lambda_i\big[\bar w_i - \bar p\cdot x_i\big]=0, \end{cases} \tag{6.4}$$

but the positivity of $\bar\lambda_i$ converts the second part of this into the equation $\bar w_i - \bar p\cdot x_i = 0$.

In our setting, with strong survivability, conditions (E2) and (E2$^+$) are actually equivalent, as we demonstrate next.

Theorem 5 (enhancement consequence of strong survivability)**.** *Under assumption (A6), along with (A1) and (A3), every equilibrium is an equilibrium enhanced by utility scale factors, and similarly for every ε-equilibrium.*

Proof. Suppose the vectors $\bar p$, $\bar x_i$ and $\bar y_j$ give an equilibrium as in Definition 1. By (E2), $\bar x_i$ is an optimal solution to the problem of maximizing the concave function u_i over the convex set X_i subject to the constraint $\bar p\cdot x_i \leq \bar w_i$, where $\bar w_i$ is the "wealth" in (6.2). For this optimization problem of convex type, with u_i and X_i related as in (A1), if we can verify the existence of some $\hat x_i\in X_i$ with $u_i(\hat x_i) > -\infty$ such that $\bar p\cdot\hat x_i < \bar w_i$ (a type of Slater constraint qualification), we will get the existence of $\bar\lambda_i\geq 0$ such that $\bar x_i$ achieves the maximum of $u_i(x_i) - \bar\lambda_i\bar p\cdot x_i$ over all $x_i\in X_i$ and in addition satisfies $\bar p\cdot x_i\leq \bar w_i$, with equality holding unless $\bar\lambda_i = 0$. In fact we can't have $\bar\lambda_i=0$, because that would mean that the maximum of $u_i(x_i)$ itself was achieved at $\bar x_i$, contrary to the insatiability in (A3), so the conclusion we wish can be obtained in this way.

Let the vectors $\hat x_i$ and $\hat y_j^i$ have the property in (A6), moreover with $\hat x_i$ in the interior of X_i, as can be arranged through (A3). Then $u_i(\hat x_i) > -\infty$ and $\bar p\cdot\hat x_i < \hat w_i$, where

$$\hat w_i = \bar p\cdot\Big[\, e_i + \sum\nolimits_{j=1}^J \theta_{ij}\hat y_j^i\,\Big] = \bar p\cdot e_i + \sum\nolimits_{j=1}^J \theta_{ij}\bar p\cdot\hat y_j^i. \tag{6.5}$$

We have $\bar p\cdot\hat y_j^i\leq \bar p\cdot\bar y_j$ for all j by (E3). Since the coefficients θ_{ij} are nonnegative, this implies $\hat w_i\leq\bar w_i$, so that $\bar p\cdot\hat x_i < \bar w_i$ as desired. □

Thanks to Theorem 5, we will get more from our proof of Theorem 3 than just a classical Walrasian equilibrium. An equilibrium enhanced by utility scale factors will be an automatic by-product. This is a new contribution. Obtaining such an enhanced equilibrium never seems to have been considered in the economics literature. No doubt that's because this notion depends heavily on having preferences expressed by concave utilities (even quasiconcave utilities wouldn't be enough), and economists had their eyes turned elsewhere.

We regard the utility functions u_i from now on as being defined on all of $\mathbb{R}^K$ by adopting the convention that

$$u_i(x_i) = -\infty \quad \text{when } x_i\notin X_i. \tag{6.6}$$

Then $-u_i$ is a lower semicontinuous, proper, convex function such that

$$q_i\in\partial[-u_i](\bar x_i) \iff \bar x_i\in\operatorname*{argmax}_{x_i\in X_i}\big\{\, u_i(x_i) + q_i\cdot x_i\big\}. \tag{6.7}$$

Theorem 6 (variational inequality for enhanced equilibrium)**.** *The elements $\bar p$, $\{\bar x_i\}_{i=1}^I$, $\{\bar\lambda_i\}_{i=1}^I$ and $\{\bar y_j\}_{j=1}^J$ furnish a Walrasian equilibrium enhanced by utility scale factors if and only if, when strung out as a vector $(\bar p;\ldots,\bar x_i,\ldots;\ldots,\bar\lambda_i,\ldots;\ldots,\bar y_j,\ldots)$, they solve the functional variational inequality problem* VI(f,F) *for the lower semicontinuous, proper, convex function*

$$\begin{aligned} &f\big(p;\ldots,x_i,\ldots;\ldots,\lambda_i\ldots;\ldots,y_j,\ldots\big) \\ &\qquad = \delta_{\mathbb{R}_+^K}(p) + \textstyle\sum_{i=1}^I[-u_i](x_i) + \sum_{i=1}^I \delta_{\mathbb{R}_+}(\lambda_i) + \sum_{j=1}^J \delta_{Y_j}(y_j) \end{aligned} \tag{6.8}$$

and the continuous mapping

$$\begin{aligned} &F\big(p;\dots,x_i,\dots;\dots,\lambda_i\dots;\dots,y_j,\dots\big)\\ &\quad= \Big(\Sigma_{i=1}^I(e_i-x_i)+\Sigma_{j=1}^J y_j;\dots,\lambda_i p,\dots;\dots,p\cdot\big(e_i-x_i+\Sigma_{j=1}^J\theta_{ij}y_j\big),\dots;\dots,-p,\dots\Big).\end{aligned} \tag{6.9}$$

Proof. Through the subgradient expression of normal cones in (3.5), the variational inequality in question comes down to the following relations holding, component by component:

$$-\Sigma_{i=1}^I(e_i-\bar{x}_i)-\Sigma_{j=1}^J\bar{y}_j\in N_{\boldsymbol{R}_+^K}(\bar{p}), \tag{6.10}$$

$$-\bar{\lambda}_i\bar{p}\in\partial[-u_i](\bar{x}_i)\ \text{ for }\ i=1,\dots,I, \tag{6.11}$$

$$\bar{p}\cdot\big(e_i-\bar{x}_i+\Sigma_{j=1}^J\theta_{ij}\bar{y}_j\big)\in N_{\boldsymbol{R}_+}(\bar{\lambda}_i)\ \text{ for }\ i=1,\dots,I, \tag{6.12}$$

$$\bar{p}\in N_{Y_j}(\bar{y}_j)\ \text{ for }\ j=1,\dots,J. \tag{6.13}$$

Here (6.10) is the complementarity condition

$$\bar{p}\geq 0,\qquad \Sigma_{i=1}^I(e_i-\bar{x}_i)+\Sigma_{j=1}^J\bar{y}_j\geq 0,\qquad \bar{p}\cdot\Big[\Sigma_{i=1}^I(e_i-\bar{x}_i)+\Sigma_{j=1}^J\bar{y}_j\Big]=0, \tag{6.14}$$

which amounts to the price nonnegativity in (E1) along with the market clearing requirement in (E4). In view of the subgradient property in (6.7), the relations in (6.11) say that

$$\bar{x}_i\in\operatorname*{argmax}_{x_i\in X_i}\big\{\,u_i(x_i)-\bar{\lambda}_i\bar{p}\cdot x_i\big\}\ \text{ for }\ i=1,\dots,I, \tag{6.15}$$

whereas the ones in (6.12) are equivalent to the complentarity conditions

$$\bar{\lambda}_i\geq 0,\quad \bar{p}\cdot x_i-\bar{w}_i\leq 0,\quad \bar{\lambda}_i\big[\bar{p}\cdot x_i-\bar{w}_i\big]=0\quad\text{for }\ \bar{w}_i=\bar{p}\cdot\Big[e_i+\Sigma_{j=1}^J\theta_{ij}\bar{y}_j\Big]. \tag{6.16}$$

The combination of (6.15) and (6.16) is the saddle point condition (6.4). But from (6.15) it's apparent that both $\bar{\lambda}_i\neq 0$ and $\bar{p}\neq 0$, because otherwise the insatiability in (A3) would be contradicted. We therefore have $\bar{\lambda}_i>0$ as required in the definition of an enhanced equilibrium, and furthermore the price nontriviality demanded in (E1). Finally, we recall that the normal cone condition in (6.13) is equivalent to the profit maximization in (E3). □

The mapping F in Theorem 6 fails to have the monotonicity property (3.6). A compensating feature of this variational inequality representation, however, as far as computational potential is concerned, is that it's tantamount to a view of enhanced equilibrium in which all the elements except the factors $\bar{\lambda}_i$ come out of a single, large-scale optimization problem. The role of these factors is to provide weights for the individual consumers in constructing an appropriate "collective" utility function.

Theorem 7 (enhanced equilibrium as collective optimization)**.** *The elements $\bar{p}$, $\{\bar{x}_i\}_{i=1}^I$, $\{\bar{\lambda}_i\}_{i=1}^I$ and $\{\bar{y}_j\}_{j=1}^J$, with $\bar{\lambda}_i>0$, furnish an equilibrium enhanced by utility scale factors if and only if*

(a) $\{\bar{x}_i\}_{i=1}^I$ *and* $\{\bar{y}_j\}_{j=1}^J$ *solve the problem*

$$\begin{aligned}&\textit{maximize }\ U(x_1,\dots,x_I)=\textstyle\sum_{i=1}^I\bar{\lambda}_i^{-1}u_i(x_i)\ \textit{ subject to}\\ &x_i\in X_i,\ \ y_j\in Y_j,\ \ \textstyle\sum_{i=1}^I x_i-\sum_{j=1}^J y_j-\sum_{i=1}^I e_i\leq 0,\end{aligned} \tag{6.17}$$

(b) *$\bar{p}$ is a multiplier vector at optimality for the inequality constraint in this problem, and*

(c) *the budget equations $\bar{p}\cdot x_i=\bar{p}\cdot\big[\,e_i+\sum_{j=1}^J\theta_{ij}\bar{y}_j\big]$ are satisfied.*

Proof. The conditions placed on $\{\bar{x}_i\}_{i=1}^I$, $\{\bar{y}_j\}_{j=1}^J$ and $\bar{p}$ mean that these elements furnish a saddle point of the function

$$L(\dots,x_i,\dots;\dots,y_j,\dots;p)\ =\ U(\dots,x_i,\dots)-p\cdot\Big[\sum\nolimits_{i=1}^I x_i-\sum\nolimits_{j=1}^J y_j-\sum\nolimits_{i=1}^I e_i\Big]$$

relative to maximizing with respect to $x_i\in X_i$ and $y_j\in Y_j$ and minimizing with respect to $p\in I\!R_+^K$. It's easy to see from the separability of U that this comes down to (E3), (E4) and the saddle point condition (6.4) for (E2$^+$), explained after Definition 4, plus the stipulation that $\bar{p}\geq 0$ in (E1). It requires $\bar{p}\neq 0$ because of the insatiability in assumption (A3). □

Theorem 7 suggests the possibility of computational methods in which, at each iteration, weights are assigned to the consumers and then a single large-scale optimization problem is solved to determine corresponding $\bar{x}_i$, $\bar{y}_j$, and $\bar{p}$. Somehow then the weights would be updated. The issue would be how to do the updating so as to obtain convergence with the budget equations (6.1) satisfied in the limit.

The result in Theorem 7 has partly been foreshadowed in economics, but without the technical formulation which would support application to our context. In the textbook of Varian [56, pp. 333–335], for example the idea of using the weights $1/\lambda_i$ to get a "social welfare function" is suggested for a pure exchange economy. The context, however, is one of assuming the differentiability of utility functions, neglecting boundary complications, and even the differentiability of some functions arising secondarily from optimization, namely so-called indirect utility functions, none of which ought to be expected, much less taken for granted.

7. Truncation and Existence Proof The task we finally take up now is that of replacing the variational inequality VI(f, F) in Theorem 7 by a modified variational inequality VI$(\hat{f}, F)$ which can be confirmed to have the same solutions (subject to price normalization), but enjoys the important feature that $\operatorname{dom} \hat{f}$ is bounded.

Proposition 5 (truncation of consumption and production). *There exist $b \in I\!R^K$ and $\beta \in I\!R$ such that*

$$\sum_{i=1}^I x_i \le \sum_{i=1}^I e_i + \sum_{j=1}^J y_j \text{ with } x_i \in X_i,\ y_j \in Y_j \quad \Longrightarrow \quad \begin{cases} 0 \le x_i < b \text{ for all } i, \\ \sum_{j=1}^J ||y_j||_\infty < \beta. \end{cases} \tag{7.1}$$

Moreover such b and β can be obtained as follows. Take $\hat{p}$ as in (A5) and choose $\varepsilon > 0$ and $\alpha_j \in I\!R$ such that

$$||p - \hat{p}||_1 \le \varepsilon \quad \Longrightarrow \quad \sup_{y_j \in Y_j} p \cdot y_j \le \alpha_j \text{ for all } j, \tag{7.2}$$

as can be done on the basis of Proposition 3. Then take any $\beta > \varepsilon^{-1}\big[\hat{p} \cdot \sum_{i=1}^I e_i + \sum_{j=1}^J \alpha_j\big]$ and any b such that $||x||_\infty < ||\sum_{i=1}^I e_i||_\infty + \beta$ implies $x < b$.

Proof. It's possible to choose ε and α_j as described, because, according to Proposition 3, $\hat{p}$ is interior to the effective domain of the (convex) support function $S_j(p) = \sup\{\, p \cdot y_j \,|\, y_j \in Y_j\}$; for instance one can choose α_j to be the largest of the values of $S_j(p)$ for the vectors p that are the vertices of the polyhedron $\{\, p \,|\, ||p - \hat{p}||_1 \le \varepsilon\}$. Then

$$y_j \in Y_j \quad \Longrightarrow \quad \alpha_j \ge \sup_{||q||_1 \le \varepsilon} (\hat{p} + q) \cdot y_j = \hat{p} \cdot y_j + \varepsilon ||y_j||_\infty \tag{7.3}$$

Suppose now that the vectors x_i and y_i satisfy the condition on the left in (7.1). Because $x_i \ge 0$ by (A1), we must have $\sum_{j=1}^J y_j \ge -\sum_{i=1}^I e_i$, so that $\sum_{j=1}^J \hat{p} \cdot y_j \ge -\hat{p} \cdot \sum_{i=1}^I e_i$. Combining this with the inequality in (7.3), we obtain $\varepsilon \sum_{j=1}^J ||y_j||_\infty \le \hat{p} \cdot \sum_{i=1}^I e_i + \sum_{j=1}^J \alpha_j$, which yields, for β as specified, the norm estimate in (7.1).

Returning to the inequality on the left of (7.1), with every $x_i \ge 0$, we see that each x_{i_0} by itself must in particular satisfy $0 \le x_{i_0} \le \sum_{i=1}^I e_i + \sum_{j=1}^J y_j$, hence $||x_{i_0}||_\infty \le ||\sum_{i=1}^I e_i||_\infty + \sum_{j=1}^J ||y_j||_\infty$, where the final term is less than β. This implies, for b as specified, that $x_{i_0} < b$. □

The proof of the next statement, like that of Proposition 4, makes essential use of the strong survivability assumption (A6) as furnishing a Slater condition on the budget constraints. We also appeal to the fact that the price vectors $\bar{p}$ in an equilibrium can harmlessly be scaled to lie in the price simplex P. We work with a saddle point condition which can ultimately be related to the one connected with the utility scale factors in an enhanced equilibrium, but operates without restriction in advance to an equilibrium price vector $\bar{p}$.

Proposition 6 (truncation of utility scale factors). *Choose vectors $\hat{x}_i \in X_i$ and $\hat{y}_j^i \in Y_j$ such that*

$$u_i(\hat{x}_i) > -\infty \text{ and } \hat{x}_i < \hat{e}_i \text{ for } \hat{e}_i = e_i + \sum\nolimits_{j=1}^J \theta_{ij} \hat{y}_j, \tag{7.4}$$

as is possible through (A6) combined with (A3). Take $\varepsilon_i > 0$ small enough that $\hat{x}_{ik} + \varepsilon_i < \hat{e}_{ik}$ for $k = 1, \dots, K$. Choose any $\hat{b} \in I\!R^K$ such that

$$\hat{b} > \hat{x}_i \ \text{ for } \ i = 1, \dots, I, \tag{7.5}$$

and set

$$\hat{\lambda}_i = [u(\hat{b}) - u(\hat{x}_i)]/\varepsilon_i. \tag{7.6}$$

Then for any $p \in P$ and $w_i \geq p \cdot \hat{e}_i$, the saddle points $(\bar{x}_i, \bar{\lambda}_i)$ of

$$L_i(x_i, \lambda_i) = u_i(x_i) + \lambda_i\big[w_i - p \cdot x_i\big] \tag{7.7}$$

in minimizing with respect to $x_i \in X_i$, $x_i \leq \hat{b}$, and maximizing with respect to $\lambda_i \in [0, \infty)$ are the same as those obtained from minimizing with respect to $x_i \in X_i$, $x_i \leq \hat{b}$, and maximizing with respect to $\lambda_i \in [0, \hat{\lambda}_i]$. Moreover, all such saddle points actually have

$$\bar{\lambda}_i < \hat{\lambda}_i \ \ \text{ for } \ i = 1, \dots, I. \tag{7.8}$$

In particular, as long as $\hat{b} \geq b$ for the vector b in Proposition 5, this bound must be satisfied by the utility scale factors $\bar{\lambda}_i$ in any enhanced equilibrium with $\bar{p} \in P$.

Proof. The truth of the final claim will follow from that of the general claim, inasmuch as the factors $\bar{\lambda}_i$ have already been seen to correspond to saddle points of the Lagrangian in (6.2) for the wealth value $\bar{w}_i$ in (6.3). Here we recall that the vectors $\bar{x}_i$ in an enhanced equilibrium have to satisfy $\bar{x}_i < b$ for the b in Proposition 5. The choice of $\hat{b} \geq b$ ensures therefore that the previous saddle point condition implies the present saddle point condition in the case of $\lambda_i \in [0, \infty)$.

To prove the general claim, we draw on basic duality theory in convex optimization, which characterizes saddle points of the concave-convex function L_i in terms of primal and dual problems of optimization [49]. Let

$$\varphi_i(\lambda_i) = \sup\big\{\, L_i(x_i, \lambda_i) \,\big|\, x_i \in X_i,\ x_i \leq \hat{b} \big\}, \tag{7.9}$$

noting that φ_i is a finite convex function on $[0, \infty)$ which is lower semicontinuous (actually continuous). For a saddle point of the first kind, with $\lambda_i \in [0, \infty)$, the dual problem is to

$$\text{minimize } \ \varphi_i(\lambda_i) \ \text{ with respect to } \ \lambda_i \in [0, \infty), \tag{7.10}$$

whereas the primal problem is to

$$\text{maximize } \ \inf_{\lambda_i \in [0,\infty)} L_i(x_i, \lambda_i) \ \text{ with respect to } \ x_i \in X_i,\ x_i \leq \hat{b}. \tag{7.11}$$

For a saddle point of the second kind, with $\lambda_i \in [0, \hat{\lambda}_i]$, the dual problem is to

$$\text{minimize } \ \varphi_i(\lambda_i) \ \text{ with respect to } \ \lambda_i \in [0, \hat{\lambda}_i]. \tag{7.12}$$

whereas the primal problem is to

$$\text{maximize } \ \inf_{\lambda_i \in [0,\hat{\lambda}_i]} L_i(x_i, \lambda_i) \ \text{ with respect to } \ x_i \in X_i,\ x_i \leq \hat{b}. \tag{7.13}$$

The primal problems don't have to concern us beyond the observation that in both cases an optimal solution exists because of the closedness of X_i and upper semicontinuity of u_i in (A1) and (A2), along with the boundedness coming from having $X_i \subset I\!R^K_+$ and $x_i \leq \hat{b}$. (The choice of $\hat{b}$ and w_i ensures that the constraints in (7.11) can be satisfied.) All we really need is the following fact of duality theory ([49], [47, Sec. 30], or [52, 11H]): if in either dual problem there is an α such that the set of feasible λ_i satisfying $\varphi_i(\lambda_i) \leq \alpha$ is nonempty and bounded, then the solutions to that dual problem are precisely the values $\bar{\lambda}_i$ paired with some $\bar{x}_i$ in the saddle point condition associated with that problem.

The boundedness property in question for φ_i holds trivially in (7.12), so our task reduces to demonstrating that it holds in (7.10) as well, and that the optimal solutions $\bar{\lambda}_i$ problems (7.10) and (7.12) are the same and lie in $[0, \hat{\lambda}_i)$. On the one hand, we have

$$\varphi_i(0) = \sup_{x_i \leq \hat{b}} u_i(x_i) = u_i(\hat{b}) < \infty \tag{7.14}$$

because of (A3). On the other hand, we have

$$\varphi_i(\lambda_i) \geq L_i(\hat{x}_i, \lambda_i) = u_i(\hat{x}_i) + \lambda_i\big[w_i - p \cdot \hat{x}_i\big] \ \text{ for all } \ \lambda_i \geq 0,$$

where the conditions imposed on w_i and ε_i guarantee, through having $p \in P$, that

$$w_i - p\cdot\hat{x}_i \;\geq\; p\cdot\hat{e}_i - p\cdot\hat{x}_i = p\cdot[\hat{e}_i - \hat{x}_i] \;>\; \varepsilon_i.$$

Hence $\varphi_i(\lambda_i) > u_i(\hat{x}_i) + \lambda_i\varepsilon_i$ when $\lambda_i > 0$, so that

$$\lambda_i < [\varphi_i(\lambda_i) - u_i(\hat{x}_i)]/\varepsilon_i \;\text{ for all }\; \lambda_i > 0. \tag{7.15}$$

In view of the definition of $\hat{\lambda}_i$ in (7.5), we obtain from (7.14) and (7.15) that

$$\emptyset \neq \big\{\, \lambda_i \geq 0 \,\big|\, \varphi_i(\lambda_i) \leq \varphi_i(0) \big\} \subset [0, \hat{\lambda}_i).$$

This furnishes the required boundedness property and confirms that all optimal solutions to the dual problems (7.10) and (7.12) must lie in $[0, \hat{\lambda}_i)$. □

Theorem 8 (truncated variational inequality for enhanced equilibrium)**.** *Let b and β be as in Proposition 5. Take $\hat{b}$ and $\hat{\lambda}_i$ as in Proposition 6 with respect to $\hat{x}_i$ and $\hat{y}^i_j$ obtained from (A6), but also with $\hat{b} \geq b$. Choose $\hat{\beta} \geq \beta$ such that $\hat{\beta} \geq ||y^i_j||_\infty$ for all i, j. Define*

$$\hat{Y}_j = \Big\{\, y_j \in Y_j \,\Big|\, ||y_j||_\infty \leq \hat{\beta} \Big\}, \qquad \hat{u}_i = \begin{cases} u_i(x_i) & \text{if } x_i \in X_i \text{ and } x \leq \hat{b}, \\ -\infty & \text{otherwise,} \end{cases} \tag{7.16}$$

and let

$$\begin{aligned} &\hat{f}\,(p;\ldots,x_i,\ldots;\ldots,\lambda_i\ldots;\ldots,y_j,\ldots) \\ &\qquad = \delta_P(p) + \textstyle\sum_{i=1}^I [-\hat{u}_i](x_i) + \sum_{i=1}^I \delta_{[0,\hat{\lambda}_i]}(\lambda_i) + \sum_{j=1}^J \delta_{\hat{Y}_j}(y_j). \end{aligned} \tag{7.17}$$

Take the mapping F as in Theorem 6. The solutions to $\mathrm{VI}(\hat{f}, F)$ *are then the same as the solutions to the variational inequality* $\mathrm{VI}(f, F)$ *in Theorem 6 (which constitute an enhanced equilibrium), except for the price vector being normalized to belong to the simplex P.*

Proof. This is built primarily on Propositions 5 and 6, but we also need to confirm that the replacement of the orthant $\mathbb{R}^K_+$ by the simplex P doesn't give difficulty. The difference caused by this replacement is that, instead of getting $\bar{z} \in N_{\mathbb{R}^K_+}(\bar{p})$, which corresponds to the market clearing conditions (2.4) in terms of the excess demand vector $\bar{z}$, we only have $\bar{z} \in N_P(\bar{p})$. That means

$$\bar{p}_k = 0 \text{ for any good } k \text{ with } \bar{z}_k < \zeta, \text{ where } \zeta = \max\{\bar{z}_1, \ldots, \bar{z}_K\}. \tag{7.18}$$

In any enhanced equilibrium, we have (7.18) holding with $\zeta = 0$. Furthermore, $\bar{x}_i$ and $\bar{y}_j$ have to satisfy the bounds in Proposition 5, and therefore $\bar{x}_i < \hat{b}$ and $||\bar{y}_j||_\infty < \hat{\beta}$. At the same time, we have $\bar{\lambda}_i < \hat{\lambda}_i$ holding by Proposition 6. Because of these strict bounds, the conditions in $\mathrm{VI}(\hat{f}, F)$ reduce to the same thing as the conditions in $\mathrm{VI}(f, F)$.

Conversely, now, suppose that $\bar{p}$, $\{\bar{x}_i\}_{i=1}^I$, $\{\bar{\lambda}_i\}_{i=1}^I$ and $\{\bar{y}_j\}_{j=1}^J$ satisfy the conditions comprising $\mathrm{VI}(\hat{f}, F)$, i.e., we have $\bar{p} \in P$ and (7.18), along with

$$\bar{y}_j \in \operatorname*{argmax}_{y_j \in \hat{Y}_j} \bar{p}\cdot y_j, \tag{7.19}$$

and $(\bar{x}_i, \bar{\lambda}_i)$ being a saddle point of the Lagrangian

$$\bar{L}_i(x_i, \lambda_i) = u_i(x_i) + \lambda_i\big[\bar{w}_i - \bar{p}\cdot x_i\big]$$

in maximizing with respect to $x_i \in \hat{X}_i$ and minimizing with respect to $\lambda_i \in [0, \hat{\lambda}_i]$, where $\bar{w}_i$ is the value in (6.3). By Proposition 6, $(\bar{x}_i, \bar{\lambda}_i)$ is also a saddle point of $\bar{L}_i$ for maximizing with respect to $x_i \in \hat{X}_i$ and minimizing with respect to $\lambda_i \in [0, \infty)$. That entails $\bar{x}_i$ giving the maximum of $u_i(x_i)$ subject to $x_i \in \hat{X}_i$ and the budget constraint $\bar{p}\cdot x_i \leq \bar{w}_i$. In particular, then, through (6.3) and (2.1) we have

$$\begin{aligned} 0 \geq \textstyle\sum_{i=1}^I \big[\bar{p}\cdot\bar{x}_i - \bar{w}_i\big] &= \bar{p}\cdot\textstyle\sum_{i=1}^I \Big[\bar{x}_i - \sum_{j=1}^J \theta_{ij}\bar{y}_j - e_i\Big] \\ &= \bar{p}\cdot\Big[\textstyle\sum_{i=1}^I \bar{x}_i - \sum_{j=1}^J \bar{y}_j - \sum_{i=1}^I e_i\Big] = \bar{p}\cdot\bar{z}, \end{aligned} \tag{7.20}$$

which implies that $\zeta \leq 0$ in (7.18). That in turn tells us that $\bar{z} \leq 0$, or in other words, that $\sum_{i=1}^I \bar{x}_i - \sum_{j=1}^J \bar{y}_j - \sum_{i=1}^I e_i \leq 0$. Then by Proposition 5 and the choice of $\hat{b}$ and $\hat{\beta}$ we necessarily have $\bar{x}_i < \hat{b}$ and $||\bar{y}_j||_\infty < \hat{\beta}$. Therefore, (E2) and (E3) hold: $\bar{x}_i$ actually furnishes the maximum of $u_i(x_i)$ subject to the budget constraint $\bar{p}\cdot x_i \leq \bar{w}_i$ and $x_i \in X_i$, not just $x_i \in \hat{X}_i$, and $\bar{y}_j$ belongs in (7.19) to the maximum of $\bar{p}\cdot y_j$ over $y_j \in Y_j$, not just $y_j \in \hat{Y}_j$. This implies further through Proposition 4 that ($\mathrm{E2}^+$) holds, so $\bar{\lambda}_i > 0$ and $\bar{p}\cdot\bar{x}_i - \bar{w}_i = 0$ for all i. Going back to (7.20), we see that then $\bar{p}\cdot\bar{z} = 0$. This confirms (E4), and we conclude that we do have an enhanced equilibrium. □

Theorem 9 (existence of enhanced Walrasian equilibrium)**.** *Under (A1)–(A6), a Walrasian equilibrium exists which is moreover an enhanced equilibrium in the sense of Definition 4. It can be determined by solving the functional variational inequality* $\mathrm{VI}(\hat{f}, F)$ *in Theorem 8.*

Proof. In the variational inequality $\mathrm{VI}(\hat{f}, F)$, we have $\operatorname{dom}\hat{f}$ bounded, so a solution exists by Theorem 1. According to Theorem 8, that solution also works for the variational inequality $\mathrm{VI}(f, F)$ in Theorem 6 and therefore actually furnishes an enhanced Walrasian equilibrium. □

Theorem 10 (existence of enhanced Walrasian ε-equilibrium)**.** *Under (A1)–(A5), but with (A6) replaced by the combination of the survivability in (A6⁻) with the direct assumption of strict economic feasibility, an enhanced Walrasian* ε*-equilibrium exists for any* $\varepsilon > 0$.

Proof. We can restrict our attention to price vectors in the simplex P. The assumptions replacing (A6) guarantee for any $\delta > 0$ the existence of modified endowments e'_i satisfying

$$||e'_i - e_i||_\infty < \delta, \qquad \sum\nolimits_{i=1}^{I} e'_i = \sum\nolimits_{i=1}^{I} e_i,$$

such that (A6) would hold if each e_i were replaced by e'_i. Theorem 9 is applicable to that modified model. It remains only to set $\pi_i = \bar{p} \cdot [e'_i - e_i]$ and observe that the conditions on these values in Definition 3 will then be fulfilled for ε, as long as δ is small enough. □

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