Contextfreeness in Symbolic Dynamics

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Subshifts.

Let Σ be a finite alphabet. By a subshift $X \subset \Sigma^{\mathbb{Z}}$ is meant a closed subset of $\Sigma^{\mathbb{Z}}$ that is invariant under the shift S,

$$S(x_i)_{i\in\mathbb{Z}} = (x_{i+1})_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

A word is called admissible for a subshift if it appears in some point of the subshift. We denote the language of admissible words of a subshift $X \subset \Sigma^{\mathbb{Z}}$ by $\mathcal{L}(X)$.

Notation for subshifts.

Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set for $a \in \mathcal{L}(X)$,

$$\Gamma^{+}(a) = \{b \in \mathcal{L}(X) : ab \in \mathcal{L}(X)\}.$$

Γ^{-} has the symmetric meaning. With

$$X_{[1,\infty)} = \{(x_i)_{1 \leq i < \infty} : x \in X\}$$

we also set

$$\Gamma^{+}_{\infty}(a) = \{x^{+} \in X_{[1,\infty)} : ax^{+} \in X_{[1,\infty)}\},$$

and

$$\omega^{-}(a) = \bigcap_{x^{+} \in \Gamma^{+}_{\infty}(a)} \{x^{-} \in \Gamma^{-}_{\infty}(a) : x^{-}ax^{+} \in X\}.$$

Semisynchronization.

A word $v \in \mathcal{L}(X)$ is called synchronizing for a subshift $X \subset \Sigma^{\mathbb{Z}}$ if for $u, w \in \mathcal{L}(X)$, $uv, vw \in \mathcal{L}(X)$ implies $uvw \in \mathcal{L}(X)$. A topologically transitive subshift with a synchronizing word is called synchronizing.

A word $v \in \mathcal{L}(X)$ is called semisynchronizing for a subshift $X \subset \Sigma^{\mathbb{Z}}$ if there is a transitive point in $\omega^-(v)$. A subshift is called semisynchronizing if it has a semisynchronizing word. A semisynchronizing subshift is called standard semisynchronizing if for all $a \in \mathcal{L}(X)$ there exists an $x^- \in \Gamma^-_\infty(a)$ such that for all $b \in \Gamma^+(a), x^- \notin \omega^-(ab)$.

Synchronization, semisynchronization and standard semisynchronization are invariants of topological conjugacy. Here we consider standard semisynchronizing, non-synchronizing subshifts.

Shannon graphs.

We will consider directed graphs and denote she source vertex of an edge by s, and the target vertex of an edge by t.
A labeled directed graph $(\mathcal{V}, \mathcal{E}, \lambda)$ with labeling alphabet Σ is called a Shannon graph if for all $V \in \mathcal{V}$ and for $\sigma \in \Sigma$ there is at most one edge that leaves V and that carries the label σ. We consider here only Shannon graphs in which every vertex has a finite number of incoming edges.
We extend the label map to paths in the graph by concatenation. We say that a Shannon graph presents a subshift $X \subset \Sigma^{\mathbb{Z}}$ if $\mathcal{L}(X)$ coincides with the labels of the finite non-empty paths in the graph.

The semisynchronizing Shannon graph of a semisynchronizing subshift.

If v is a semisynchronizing word of a subshift, then for all $\sigma \in \Gamma^+(v), v\sigma$ is also semisynchronizing. It follows that a semisynchronizing subshift gives rise to its semisynchronizing Shannon graph, that has as vertex sets the sets $\Gamma^+_\infty(v)$, v a semisynchronizing word of X, and where there is an edge leaving a vertex V that carries the label σ, if and only if there is a right-infinite sequence in V that starts with σ and the target vertex of this edge is the set

$$\{x^+_{(1,\infty)} : x^+_{[1,\infty)} \in V : x^+_1 = \sigma\}.$$

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A semisynchronizing subshift is presented by its semisynchronizing Shannon graph.

Strong shift equivalence of Shannon graphs.

Call two Shannon graphs $\mathcal{G}(\mathcal{V},\mathcal{E},\lambda)$ and $\tilde{\mathcal{G}}(\tilde{\mathcal{V}},\tilde{\mathcal{E}},\tilde{\lambda})$ strong shift equivalent if they can be connected by a chain $\mathcal{G}_m, 1 \leq m \leq M, M \in \mathbb{N}$, of Shannon graphs, $\mathcal{G}_1 = \mathcal{G}$, $\mathcal{G}_M = \tilde{\mathcal{G}}$, such that $\mathcal{G}_m$, and $\mathcal{G}_{m+1}$, are bipartitely related, $1 \leq m < M$. The semisynchronizing Shannon graphs of topologically conjugate semisynchronizing subshifts are strong shift equivalent.

Notation for Shannon graphs.

Given an Shannon graph $\mathcal{G}(\mathcal{V},\mathcal{E},\lambda)$, a vertex $V \in \mathcal{V}$, and a finite set $\mathcal{A} \subset \mathcal{V}$, we denote by $\Delta(V,\mathcal{A})$ the minimal length of a path in $\mathcal{G}$ that starts at V and ends in $\mathcal{A}$, and we set

$$\mathcal{S}_{\mathcal{A}}(K) = \{V \in \mathcal{V} : \Delta(V,\mathcal{A}) \leq K\}, \quad K \in \mathbb{N},$$

and

$$\mathcal{S}^{\circ}_{\mathcal{A}}(K) = \{V \in \mathcal{V} : \Delta(V,\mathcal{A}) = K\}, \quad K \in \mathbb{N}.$$

Subgraphs of Shannon graphs.

Given a Shannon graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \lambda)$ and a set $\mathcal{V}_\circ \subset \mathcal{V}$ we denote by $\mathcal{G}_{\mathcal{V}_\circ}$ the Shannon graph with vertex set $\mathcal{V}_\circ$, and edge set

$$\mathcal{E}_{\mathcal{V}_\circ} = \{ e \in \mathcal{E} : s(e), t(e) \in \mathcal{V}_\circ \},$$

that has as labeling map the restriction of the labeling map to $\mathcal{E}_{\mathcal{V}_\circ}$.

A lemma

We denote for $m \in \mathbb{N}$, E and $K > M$,$V \in \mathcal{S}^{\circ}_{\mathcal{A}}(K)$, by $\mathcal{T}(V, M)$ the set of final vertices of the paths in of length M that start at V and that approach $\mathcal{A}$ strictly.

Lemma.

Let $\mathcal{G}(\mathcal{V}, \mathcal{E}, \lambda)$ be an irreducible Shannon graph. Let $\mathcal{B} \subset \mathcal{V}$ be a finite set, and let $M \in \mathbb{N}, K_\circ \geq M$, such that for $K \geq K_\circ$ and $V, V' \in \mathcal{S}$ one has the equality

$$\mathcal{T}_{\mathcal{B}}(V, M) = \mathcal{T}_{\mathcal{B}}(V', M).$$

Then for all finite sets $\mathcal{A} \subset \mathcal{V}$ there exist $Q_\circ \in \mathbb{N}$ and $R \in \mathbb{Z}$ such that for $Q \geq Q_\circ$,

$$\mathcal{S}^{\circ}_{\mathcal{A}}(Q) = \mathcal{S}^{\circ}_{\mathcal{B}}(Q + R).$$

H1.

We say that a Shannon graph satisfies Hypothesis H1, if for a finite set $\mathcal{A} \subset \mathcal{V}$ (and therefore by the Lemma, for every finite set $\mathcal{A} \subset \mathcal{V}$) there exist $M \in \mathbb{N}, K_\circ \geq M$, such that one has for $K \geq K_\circ$ and $V, V' \in \mathcal{S}$ the equality

$$\mathcal{T}_{\mathcal{A}}(V, M) = \mathcal{T}_{\mathcal{A}}(V', M).$$

Hypothesis H1 is an invariant of the strong shift equivalence of Shannon graphs.

Approach from infinity

Given a Shannon graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \lambda)$, that satisfies hypothesis H1, and a finite set $\mathcal{A} \subset \mathcal{V}$ and $K \in \mathbb{N}$ and a vertex $V \in \mathcal{S}^{\circ}_{\mathcal{A}}(K)$, we say that V can be approached from infinity, if there exists an infinite path in $\mathcal{V} \setminus \mathcal{S}_{\mathcal{A}}(K-1)$ that ends at V. We denote the set of vertices that can be approached from infinity by $\mathcal{V}_{\infty}$.

H2.

We say that a Shannon graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \lambda)$, that satisfies Hypothesis H1, satisfies Hypothesis H2, if for a finite set $\mathcal{A} \subset \mathcal{V}$ (and therefore by the Lemma, for every finite set $\mathcal{A} \subset \mathcal{V}$) there exist $K_\circ, Q \in \mathbb{N}$ such that in every connected component $\mathcal{W}$ of $\mathcal{V}_\infty \setminus \mathcal{S}_\mathcal{A}(K-1)$ there is a path from every vertex in $\mathcal{W} \cap \mathcal{S}^\circ_\mathcal{A}(K+Q)$ to every vertex in $\mathcal{W} \cap \mathcal{S}^\circ_\mathcal{A}(K)$. Hypothesis H2 is an invariant of the strong shift equivalence of Shannon graphs.

Context-free Shannon graphs.

We say that a Shannon graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \lambda)$ is context-free, if it satisfies Hypothesis H1 and Hypothesis H2, and if there are finitely many isomorphism types among the pairs that consist of a connected component $\mathcal{W}$ of $\mathcal{V}_\infty \setminus \mathcal{S}_\mathcal{A}(K-1)$ and its boundary $\mathcal{W} \cap \mathcal{S}^\circ_\mathcal{A}(K)$, $K \in \mathbb{N}$ (see Muller and Schupp, Bull. AMS 19). Context-freeness is is an invariant of the strong shift equivalence of Shannon graphs.

Hypothesis 3.

In view of the construction of the push-down automaton that is isomorphic to the context-free Shannon graph denote by Ξ the set of isomorphism types that appear infinitely often among the pairs $(\mathcal{W}, \mathcal{W} \cap \mathcal{S}^{\circ}_{\mathcal{A}}(K)), K \in \mathbb{N}$. Also let for $\xi, \xi' \in \Xi$, $A(\xi, \xi')$ be the number of isomorphism types of embeddings of connected components with boundary in $\mathcal{S}^{\circ}_{\mathcal{A}}(K+1)$ as subgraphs connected components with boundary in $\mathcal{S}^{\circ}_{\mathcal{A}}(K)$. We call the topological Markov chain with state space Ξ and transition matrix A the stack topological Markov chain of the Shannon graph. We say that the Shannon graph satisfies hypothesis H3 if its stack topological Markov chain is irreducible. Hypothesis 3 is an invariant of the strong shift equivalence of Shannon graphs.

Hypothesis 4.

The subshift whose admissible words are the label sequences of finite paths in the Shannon graph whose target vertex is in the boundary of a connected component with an isomorphism type in Ξ, we call the stack shift of the Shannon graph. We say that a Shannon graph that satisfies hypothesis 3, satisfies hypothesis 4, if there is a $\xi_\circ \in \Xi$ and a word v of length $2l$, $l \in \mathbb{N}$ that is admissible for the stack shift, such that for a path $(b_i)_{1 \leq i \leq 2l. l \in \mathbb{N}}$ with label sequence v $t(b_i)_{1 \leq i \leq l}$ is necessarily in the boundary of a connected componen twith isomorphism type $\xi_\circ$. For a Shannon graph that satisfies hypothesis 4 the stack shift is sofic. Hypothesis 4 is an invariant of the strong shift equivalence of Shannon graphs.

A theorem.

We will not describe here the construction of the finite control of the push-down automaton, beyond saying that the finite control of the automaton also contains information on a distinguished vertex in the Shannon graph which acts as a present state, and also a description of the push-down mechanism.

Theorem.

For a standard semisynchronizing non-synchronizing subshift whose semisynchronizing Shannon graph is context-free and satisfies Hypothesis 3 and Hypothesis 4, the stack topological Markov chain is the left Fischer cover of its stack sofic shift. This left Fischer cover is an invariant of topological conjugacy.

The polycyclic monoid.

Let $N > 1$, and let $\alpha_-(n), \alpha_+(n), 0 \le n < N$, be the generators of the polycyclic monoid monoid $\mathcal{D}_N$ with the rules

$$\alpha_-(n), \alpha_+(n) = 1, \quad 1 \le n \le N,$$

$$\alpha_-(n), \alpha_+(m) = 0, \quad 1 \le n \le N, 1 \le m \le N, n \ne m.$$

(See Nivat, Perrot, Une généralisation du monoîde bicyclique, C. R. Acad. Sc. Paris (1970))

An open problem.

With the semigroup $\mathcal{D}_N^-$ ($\mathcal{D}_N^+$) that is generated by $\{\alpha_-(n) : 0 \leq n < N\}$ ($\{\alpha_+(n) : 0 \leq n < N\}$), let

$$\Sigma \subset \mathcal{D}_N^- \cup \{\mathbf{1}\} \cup \mathcal{D}_N^+,$$

be a generating set of $\mathcal{D}_N$, and let $X(\Sigma) \subset \Sigma^{\mathbb{Z}}$ be the subshift with admissible words $(\sigma_i)_{1 \leq i \leq I}$, $I \in \mathbb{N}$, given by the condition

$$\prod_{1 \leq i \leq I} \sigma_i \neq 0.$$

The subshifts $X(\Sigma)$ are standard semisynchronizing.
Problem:
Prove or disprove that the subshifts $X(\Sigma)$ are context-free semisynchronizing.