Actions of $\mathbb{Z}^k$ associated to higher rank graphs

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We construct an action of $\mathbb{Z}^k$ on a compact zero dimensional space from a higher rank graph $\Lambda$ satisfying a mild assumption generalizing the construction of the Markov shift associated to a nonnegative integer matrix. The stable Ruelle algebra $R_s(\Lambda)$ is shown to be strongly Morita equivalent to $C^*(\Lambda)$. Hence, $R_s(\Lambda)$ is simple, stable and purely infinite, if $\Lambda$ satisfies the aperiodicity condition.

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Introduction

Given a finite directed graph one constructs the associated topological Markov shift as the shift map on the space of two-sided infinite paths, a compact zero dimensional space when endowed with the natural topology.

This dynamical system is a Smale space in the sense of Ruelle. Using the theory of groupoids and the stable equivalence relation of a Smale space, Putnam constructed the stable C*-algebra $S$ and its crossed product, the Ruelle algebra $R_s$, intended as a generalization of a Cuntz-Krieger algebra.

We will show that these constructions apply to a $k$-graph (a combinatorial notion inspired by work of Robertson and Steger). There is a natural action of $\mathbb{Z}^k$ on the space of two-sided paths which enjoys all the key properties of a Smale space.

We invoke the theory of groupoid equivalence to show that the C*-algebra of a $k$-graph is strongly Morita equivalent to its Ruelle algebra.
Higher rank graphs

A pair \((\Lambda, d)\), with \(\Lambda\) a countable small category and \(d : \Lambda \to \mathbb{N}^k\) a morphism, is said to be a \(k\)-graph if the factorization property holds: for every \(\lambda \in \Lambda\) and \(m, n \in \mathbb{N}^k\) with \(d(\lambda) = m + n\), there exist unique elements \(\mu, \nu \in \Lambda\) such that

\[ \lambda = \mu \nu, \quad m = d(\mu), \quad n = d(\nu). \]

For \(n \in \mathbb{N}^k\) write \(\Lambda^n = \{ \lambda \in \Lambda : d(\lambda) = n \}\). It will be convenient to identify \(\Lambda^0\) with the objects of \(\Lambda\). Let \(r, s : \Lambda \to \Lambda^0\) denote the range and source maps.

Let \(E = (E^0, E^1)\) be a directed graph. Then the set of finite paths \(E^*\) together with the length map defines a \(1\)-graph.

Every 2-graph arises from a pair of commuting graphs with a common vertex set.

A \(k\)-graph \(\Lambda\) is said to be irreducible if for every \(u, v \in \Lambda^0\) there is an element \(\lambda : v \to u\) in \(\Lambda\) of nonzero degree.

Given a \(k\)-graph \(\Lambda\), the opposite \(\Lambda^{\text{op}}\) is also a \(k\)-graph.
Standing Hypothesis:
For each \( n \in \mathbb{N}^k \) the restrictions of \( r \) and \( s \) to \( \Lambda^n \) are surjective and finite to one.

Definition:
Let \( \Lambda \) be a \( k \)-graph. Then \( C^*(\Lambda) \) is defined to be the universal \( C^* \)-algebra generated by a family \( \{ t_\lambda : \lambda \in \Lambda \} \) of operators satisfying:

i. \( t_v, v \in \Lambda^0 \) are mutually orth. projections,

ii. \( t_{\lambda \mu} = t_\lambda t_\mu \) for \( \lambda, \mu \in \Lambda \) with \( s(\lambda) = r(\mu) \),

iii. \( t^*_\lambda t_\lambda = t_{s(\lambda)} \) for \( \lambda \in \Lambda \),

iv. for \( v \in \Lambda^0 \) and \( n \in \mathbb{N}^k \) we have

\[
t_v = \sum_{\substack{r(\lambda) = v \\
\lambda \in \Lambda^n}} t_\lambda t^*_\lambda.
\]

This generalizes the usual definition of the Cuntz-Krieger algebra of a graph.

Gauge action:
There is a canonical action \( \alpha : \mathbb{T}^k \to \text{Aut} \left( C^*(\Lambda) \right) \) such that

\[
\alpha_z(t_\lambda) = z^{d(\lambda)} t_\lambda
\]

for \( z \in \mathbb{T}^k \) and \( \lambda \in \Lambda \).
There is a natural $\mathbb{Z}^k$ action on the analog of the two-sided path space of a $k$-graph $\Lambda$.

Define a $k$-graph $\Delta$ with object space $\mathbb{Z}^k$:

$$\Delta = \{(m,n) : m, n \in \mathbb{Z}^k, m \leq n\};$$

The structure maps are given by:

$$r(m,n) = m, \quad s(m,n) = n, \quad d(m,n) = n - m,$$

$$(\ell,n) = (\ell,m)(m,n).$$

Use $\Delta$ to form the two-sided path space:

$$\Lambda^\Delta = \{x : \Delta \to \Lambda : x \text{ is a } k\text{-graph morphism}\}.$$ 

We endow $\Lambda^\Delta$ with a topology as follows: for $n \in \mathbb{Z}^k$ and $\lambda \in \Lambda$ set

$$Z(\lambda,n) = \{x \in \Lambda^\Delta : x(n,n + d(\lambda)) = \lambda\}.$$ 

The collection of all such cylinder sets forms a basis for a topology on $\Lambda^\Delta$ for which each such subset is compact. Hence, $\Lambda^\Delta$ is a zero dimensional space and if $\Lambda^0$ is finite, then $\Lambda^\Delta$ is itself compact. Note $\Lambda^\Delta$ is nonempty.
For $n \in \mathbb{Z}^k$ define a map $\sigma^n : \Lambda^\Delta \to \Lambda^\Delta$ by
\[
\sigma^n(x)(\ell, m) = x(\ell + n, m + n).
\]
Note that $\sigma^n$ is a homeomorphism for every $n \in \mathbb{Z}^k$, $\sigma^{n+m} = \sigma^n \sigma^m$ for $n, m \in \mathbb{Z}^k$ and $\sigma^0$ is the identity map. Thus $\sigma$ defines an action of $\mathbb{Z}^k$ on $\Lambda^\Delta$.

For each $0 < r < 1$ there is a metric $\rho$ defined as follows: Let $e = (1, \ldots, 1)$; set $\rho(x, y) = 1$ if $x(0) \neq y(0)$ and set $\rho(x, y) = r^{n+1}$ where
\[
n = \max\{m \in \mathbb{N} : x \in Z(y(-me, me), -me)\}
\]
if $x(0) = y(0)$ (but $x \neq y$).

Proposition:
The action is expansive, that is, there is an $\varepsilon > 0$ such that for all $x, y \in \Lambda^\Delta$, if
\[
\rho(\sigma^n(x), \sigma^n(y)) < \varepsilon
\]
for all $n$, then $x = y$.

Proof: Take $\varepsilon = r$. If $\rho(\sigma^n(x), \sigma^n(y)) < r$ for all $n \in \mathbb{Z}^k$, then
\[
x(n - e, n + e) = y(n - e, n + e)
\]
for all $n \in \mathbb{Z}^k$. Hence, $x = y$. 

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A $k$-graph $\Lambda$ is said to be primitive if there is $n \in \mathbb{N}^k$ so that for every $u, v \in \Lambda^0$ there is $\lambda \in \Lambda^n$ with $r(\lambda) = u$ and $s(\lambda) = v$.

Proposition:
If $\Lambda$ is primitive, then $\sigma$ is topologically mixing in the sense that for any two nonempty open sets $U$ and $V$ in $\Lambda^\Delta$ there is an $N \in \mathbb{Z}^k$ so that $U \cap \sigma^n(V)$ is nonempty for all $n \geq N$.

$\Lambda^\Delta$ decomposes locally into contracting and expanding directions for the shift. For $x \in \Lambda^\Delta$ there are closed subsets $E_x, F_x \subseteq \Lambda^\Delta$ such that the unit ball is homeomorphic to $E_x \times F_x$; we have

$$\rho(\sigma^e(y), \sigma^e(z)) \leq r \rho(y, z)$$

for $y, z \in E_x$ and

$$\rho(\sigma^{-e}(y), \sigma^{-e}(z)) \leq r \rho(y, z)$$

for $y, z \in F_x$.

Our $\mathbb{Z}^k$ action satisfies Ruelle’s axioms for a Smale space.
Miscellaneous details

$E_x$ and $F_x$ are defined as follows:

$E_x = \{ y \in \Lambda^\Delta : x(m, n) = y(m, n), 0 \leq m \leq n \}$

$F_x = \{ y \in \Lambda^\Delta : x(m, n) = y(m, n), m \leq n \leq 0 \}$.

If $x(0) = y(0)$ there is a unique element in $F_x \cap E_y$; this element is denoted $[x, y]$. The map $(x, y) \mapsto [x, y]$ defines a homeomorphism

$E_z \times F_z \cong Z(z(0), 0)$

for $z \in \Lambda^\Delta$.

The one-sided path space $\Lambda^\Omega$ is defined in terms of the $k$-graph:

$\Omega = \{ (m, n) : m, n \in \mathbb{N}^k, m \leq n \}$.

The map $\pi : \Lambda^\Delta \to \Lambda^\Omega$ defined by restriction is a continuous open surjection. This will be used later to construct a groupoid equivalence.

Note that $E_x \equiv \pi^{-1}(x)$. 
Perron-Frobenius and the Parry measure

Henceforth suppose that $\Lambda$ is irreducible and $\Lambda^0$ is finite. By Perron-Frobenius there exist $a : \Lambda^0 \rightarrow \mathbb{R}_+$, $b : \Lambda^0 \rightarrow \mathbb{R}_+$ and $\theta \in \mathbb{R}_+^k$ with

$$\sum_{v \in \Lambda^0} a(v)b(v) = 1$$

such that for all $p \in \mathbb{N}^k$ we have

$$\sum_{u \in \Lambda^0} a(u)|\Lambda^p|(u,v) = \theta^p a(v)$$

$$\sum_{v \in \Lambda^0} |\Lambda^p|(u,v)b(v) = \theta^p b(u)$$

where $|\Lambda^p|(u,v)$ is the number of elements $\lambda : v \rightarrow u$ of degree $p$.

This fact allows one to define the analog of the Parry measure.

Proposition:
There is a shift invariant probability measure $\mu$ on $\Lambda^\Delta$ such that

$$\mu(Z(\lambda, n)) = \theta^{-d(\lambda)}a(r(\lambda))b(s(\lambda)),$$

for all $\lambda \in \Lambda$ and $n \in \mathbb{Z}^k$. 
Consider the stable and unstable equivalence relations (cf. Putnam):

\[ x \sim_s y \text{ if } \lim_{j \to \infty} \rho(\sigma^j(x), \sigma^j(y)) = 0 \]

and

\[ x \sim_u y \text{ if } \lim_{j \to \infty} \rho(\sigma^{-j}(x), \sigma^{-j}(y)) = 0 \]

for \( x, y \in \Lambda^\Delta \).

The stable equivalence relation yields a groupoid

\[ G_s(\Lambda) = \{(x, y) \in \Lambda^\Delta \times \Lambda^\Delta : x \sim_s y\} \]

endowed with an inductive limit topology. \( G_u(\Lambda) \) is defined similarly.

The shift invariant measure \( \mu \) gives rise to Haar systems for both groupoids and we obtain the stable and the unstable C*-algebras:

\[ S(\Lambda) = C^*(G_s(\Lambda)), \]
\[ U(\Lambda) = C^*(G_u(\Lambda)). \]

Note \( U(\Lambda) = S(\Lambda^{\text{op}}) \).

Lemma:

\( S(\Lambda) \) is strongly Morita equivalent to \( C^*(\Lambda)^\alpha \)

and is therefore AF. Moreover, \( S(\Lambda) \) is simple if \( \Lambda \) is primitive.
Ruelle algebras

The shift action on $\Lambda^\Delta$ induces actions $\beta_s, \beta_u$ of $\mathbb{Z}^k$ on both $S(\Lambda)$ and $U(\Lambda)$. We define the Ruelle algebras as the crossed products:

$$R_s(\Lambda) = S(\Lambda) \times_{\beta_s} \mathbb{Z}^k,$$
$$R_u(\Lambda) = U(\Lambda) \times_{\beta_u} \mathbb{Z}^k.$$  

Note that $R_s(\Lambda) = C^*(G_s(\Lambda) \times \mathbb{Z}^k)$.

Theorem:

$R_s(\Lambda)$ is strongly Morita equivalent to $C^*(\Lambda)$. Hence, $R_s(\Lambda)$ is nuclear and in the bootstrap class for which the UCT holds. Moreover, if $\Lambda$ satisfies the aperiodicity condition, then $R_s(\Lambda)$ is simple, stable and purely infinite.

Hence, by the Kirchberg-Phillips theorem the isomorphism class of $R_s(\Lambda)$ is determined by its $K$-theory.

$\Lambda$ satisfies the aperiodicity condition if there is a path $x \in \Lambda^\Omega$ which is not eventually periodic.
Idea of proof:

Use the notion of equivalence of groupoids of Muhly, Renault and Williams to show that the associated C*-algebras are strongly Morita equivalent.

Note that $C^*(\Lambda) = C^*(G_\Lambda)$ where $G_\Lambda$ is the path groupoid of $\Lambda$; the unit space is identified with $\Lambda^\Omega$, the one-sided path space. The elements of $G_\Lambda$ are triples $(x, n, y)$ with $x, y \in \Lambda^\Omega$, $n \in \mathbb{Z}^k$ such that $\sigma^\ell x = \sigma^n y$ and $n = \ell - m$ for some $\ell, m \in \mathbb{N}^k$.

The equivalence of groupoids follows from the isomorphism:

$$G_s(\Lambda) \times \mathbb{Z}^k \cong \Lambda^\Delta \rtimes G_\Lambda \rtimes \Lambda^\Delta$$

given by

$$((x, y), n) \mapsto (x, (\pi(x), n, \pi(\sigma^n y)), \sigma^n y),$$

where $\pi : \Lambda^\Delta \to \Lambda^\Omega$ is the restriction map.
Selected references


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