Actions of $\mathbb{Z}^k$ associated to higher rank graphs

Alex Kumjian, U. Nevada, Reno &
David Pask, U. Newcastle, Australia

Aperiodic Order, Dynamical Systems,
Operator Algebras and Topology,
University of Victoria, August 2002

We construct an action of $\mathbb{Z}^k$ on a compact zero dimensional space from a higher rank graph Λ satisfying a mild assumption generalizing the construction of the Markov shift associated to a nonnegative integer matrix. The stable Ruelle algebra $R_s(\Lambda)$ is shown to be strongly Morita equivalent to $C^*(\Lambda)$. Hence, $R_s(\Lambda)$ is simple, stable and purely infinite, if Λ satisfies the aperiodicity condition.

Preprint archive: OA/0112049

Introduction

Given a finite directed graph one constructs the associated topological Markov shift as the shift map on the space of two-sided infinite paths, a compact zero dimensional space when endowed with the natural topology.

This dynamical system is a Smale space in the sense of Ruelle. Using the theory of groupoids and the stable equivalence relation of a Smale space, Putnam constructed the stable C*-algebra S and its crossed product, the Ruelle algebra R_s, intended as a generalization of a Cuntz-Krieger algebra.

We will show that these constructions apply to a k-graph (a combinatorial notion inspired by work of Robertson and Steger). There is a natural action of $\mathbb{Z}^k$ on the space of two-sided paths which enjoys all the key properties of a Smale space.

We invoke the theory of groupoid equivalence to show that the C*-algebra of a k-graph is strongly Morita equivalent to its Ruelle algebra.

Higher rank graphs

A pair (Λ, d), with Λ a countable small category and $d : \Lambda \to \mathbb{N}^k$ a morphism, is said to be a k-graph if the factorization property holds: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there exist unique elements $\mu, \nu \in \Lambda$ such that

$$\lambda = \mu\nu, \quad m = d(\mu), \quad n = d(\nu).$$

For $n \in \mathbb{N}^k$ write $\Lambda^n = \{\lambda \in \Lambda : d(\lambda) = n\}$. It will be convenient to identify Λ^0 with the objects of Λ. Let $r, s : \Lambda \to \Lambda^0$ denote the range and source maps.

Let $E = (E^0, E^1)$ be a directed graph. Then the set of finite paths E^* together with the length map defines a 1-graph.

Every 2-graph arises from a pair of commuting graphs with a common vertex set.

A k-graph Λ is said to be irreducible if for every $u, v \in \Lambda^0$ there is an element $\lambda : v \to u$ in Λ of nonzero degree.

Given a k-graph Λ, the opposite Λ^{op} is also a k-graph.

Standing Hypothesis:
For each $n \in \mathbb{N}^k$ the restrictions of r and s to Λ^n are surjective and finite to one.

Definition:
Let Λ be a k-graph. Then $C^*(\Lambda)$ is defined to be the universal C*-algebra generated by a family $\{t_\lambda : \lambda \in \Lambda\}$ of operators satisfying:

i. $t_v, v \in \Lambda^0$ are mutually orth. projections,

ii. $t_{\lambda\mu} = t_\lambda t_\mu$ for $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$,

iii. $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for $\lambda \in \Lambda$,

iv. for $v \in \Lambda^0$ and $n \in \mathbf{N}^k$ we have

$$t_v = \sum_{\substack{r(\lambda)=v \\ \lambda \in \Lambda^n}} t_\lambda t_\lambda^*.$$

This generalizes the usual definition of the Cuntz-Krieger algebra of a graph.

Gauge action:
There is a canonical action $\alpha : \mathbb{T}^k \to \mathrm{Aut}\,(C^*(\Lambda))$ such that

$$\alpha_z(t_\lambda) = z^{d(\lambda)} t_\lambda$$

for $z \in \mathbb{T}^k$ and $\lambda \in \Lambda$.

There is a natural $\mathbb{Z}^k$ action on the analog of the two-sided path space of a k-graph Λ.

Define a k-graph Δ with object space $\mathbb{Z}^k$:

$$\Delta = \{(m,n) : m,n \in \mathbb{Z}^k, m \leq n\};$$

The structure maps are given by:

$$r(m,n) = m,\ \ s(m,n) = n,\ \ d(m,n) = n - m,$$

$$(\ell, n) = (\ell, m)(m, n).$$

Use Δ to form the two-sided path space:

$$\Lambda^{\Delta} = \{x : \Delta \to \Lambda : x \text{ is a } k\text{-graph morphism}\}.$$

We endow Λ^{Δ} with a topology as follows: for $n \in \mathbb{Z}^k$ and $\lambda \in \Lambda$ set

$$Z(\lambda, n) = \{x \in \Lambda^{\Delta} : x(n, n + d(\lambda)) = \lambda\}.$$

The collection of all such cylinder sets forms a basis for a topology on Λ^{Δ} for which each such subset is compact. Hence, Λ^{Δ} is a zero dimensional space and if Λ^0 is finite, then Λ^{Δ} is itself compact. Note Λ^{Δ} is nonempty.

For $n \in \mathbb{Z}^k$ define a map $\sigma^n : \Lambda^\Delta \to \Lambda^\Delta$ by

$$\sigma^n(x)(\ell, m) = x(\ell + n, m + n).$$

Note that σ^n is a homeomorphism for every $n \in \mathbb{Z}^k$, $\sigma^{n+m} = \sigma^n \sigma^m$ for $n, m \in \mathbb{Z}^k$ and σ^0 is the identity map. Thus σ defines an action of $\mathbb{Z}^k$ on Λ^Δ.

For each $0 < r < 1$ there is a metric ρ defined as follows: Let $e = (1, \ldots, 1)$; set $\rho(x, y) = 1$ if $x(0) \neq y(0)$ and set $\rho(x, y) = r^{n+1}$ where

$$n = \max\{m \in \mathbb{N} : x \in Z(y(-me, me), -me)\}$$

if $x(0) = y(0)$ (but $x \neq y$).

Proposition:
The action is expansive, that is, there is an $\varepsilon > 0$ such that for all $x, y \in \Lambda^\Delta$, if

$$\rho(\sigma^n(x), \sigma^n(y)) < \varepsilon$$

for all n, then $x = y$.

Proof: Take $\varepsilon = r$. If $\rho(\sigma^n(x), \sigma^n(y)) < r$ for all $n \in \mathbb{Z}^k$, then

$$x(n - e, n + e) = y(n - e, n + e)$$

for all $n \in \mathbb{Z}^k$. Hence, $x = y$.

A k-graph Λ is said to be primitive if there is $n \in \mathbb{N}^k$ so that for every $u, v \in \Lambda^0$ there is $\lambda \in \Lambda^n$ with $r(\lambda) = u$ and $s(\lambda) = v$.

Proposition:
If Λ is primitive, then σ is topologically mixing in the sense that for any two nonempty open sets U and V in Λ^Δ there is an $N \in \mathbb{Z}^k$ so that $U \cap \sigma^n(V)$ is nonempty for all $n \geq N$.

Λ^Δ decomposes locally into contracting and expanding directions for the shift. For $x \in \Lambda^\Delta$ there are closed subsets $E_x, F_x \subset \Lambda^\Delta$ such that the unit ball is homeomorphic to $E_x \times F_x$; we have

$$\rho(\sigma^e(y), \sigma^e(z)) \leq r\rho(y, z)$$

for $y, z \in E_x$ and

$$\rho(\sigma^{-e}(y), \sigma^{-e}(z)) \leq r\rho(y, z)$$

for $y, z \in F_x$.

Our $\mathbb{Z}^k$ action satisfies Ruelle's axioms for a Smale space.

Miscellaneous details

E_x and F_x are defined as follows:

$$E_x = \{y \in \Lambda^{\Delta} : x(m,n) = y(m,n), 0 \le m \le n\}$$
$$F_x = \{y \in \Lambda^{\Delta} : x(m,n) = y(m,n), m \le n \le 0\}.$$

If $x(0) = y(0)$ there is a unique element in $F_x \cap E_y$; this element is denoted $[x,y]$. The map $(x,y) \mapsto [x,y]$ defines a homeomorphism

$$E_z \times F_z \cong Z(z(0),0)$$

for $z \in \Lambda^{\Delta}$.

The one-sided path space Λ^{Ω} is defined in terms of the k-graph:

$$\Omega = \{(m,n) : m,n \in \mathbb{N}^k, m \le n\}.$$

The map $\pi : \Lambda^{\Delta} \to \Lambda^{\Omega}$ defined by restriction is a continuous open surjection. This will be used later to construct a groupoid equivalence.

Note that $E_x = \pi^{-1}(x)$.

Perron-Frobenius and the Parry measure

Henceforth suppose that Λ is irreducible and Λ^0 is finite. By Perron-Frobenius there exist $a : \Lambda^0 \to \mathbb{R}_+$, $b : \Lambda^0 \to \mathbb{R}_+$ and $\theta \in \mathbb{R}_+^k$ with

$$\sum_{v \in \Lambda^0} a(v)b(v) = 1$$

such that for all $p \in \mathbb{N}^k$ we have

$$\begin{aligned} \sum_{u \in \Lambda^0} a(u)|\Lambda^p|(u,v) &= \theta^p a(v) \\ \sum_{v \in \Lambda^0} |\Lambda^p|(u,v)b(v) &= \theta^p b(u) \end{aligned}$$

where $|\Lambda^p|(u,v)$ is the number of elements $\lambda : v \to u$ of degree p.

This fact allows one to define the analog of the Parry measure.

Proposition:
There is a shift invariant probability measure μ on Λ^Δ such that

$$\mu(Z(\lambda,n)) = \theta^{-d(\lambda)} a(r(\lambda)) b(s(\lambda)),$$

for all $\lambda \in \Lambda$ and $n \in \mathbb{Z}^k$.

Consider the stable and unstable equivalence relations (cf. Putnam):

$$x \sim_s y \text{ if } \lim_{j\to\infty} \rho(\sigma^{je}(x), \sigma^{je}(y)) = 0$$

and

$$x \sim_u y \text{ if } \lim_{j\to\infty} \rho(\sigma^{-je}(x), \sigma^{-je}(y)) = 0$$

for $x, y \in \Lambda^\Delta$.

The stable equivalence relation yields a groupoid

$$G_s(\Lambda) = \{(x, y) \in \Lambda^\Delta \times \Lambda^\Delta : x \sim_s y\}$$

endowed with an inductive limit topology. $G_u(\Lambda)$ is defined similarly.

The shift invariant measure μ gives rise to Haar systems for both groupoids and we obtain the stable and the unstable C*-algebras:

$$\begin{aligned} S(\Lambda) &= C^*(G_s(\Lambda)), \\ U(\Lambda) &= C^*(G_u(\Lambda)). \end{aligned}$$

Note $U(\Lambda) = S(\Lambda^{\text{op}})$.

Lemma:
$S(\Lambda)$ is strongly Morita equivalent to $C^*(\Lambda)^\alpha$ and is therefore AF. Moreover, $S(\Lambda)$ is simple if Λ is primitive.

Ruelle algebras

The shift action on Λ^{Δ} induces actions β_s, β_u of $\mathbb{Z}^k$ on both $S(\Lambda)$ and $U(\Lambda)$. We define the Ruelle algebras as the crossed products:

$$
\begin{aligned}
R_s(\Lambda) &= S(\Lambda) \times_{\beta_s} \mathbb{Z}^k, \\
R_u(\Lambda) &= U(\Lambda) \times_{\beta_u} \mathbb{Z}^k.
\end{aligned}
$$

Note that $R_s(\Lambda) = C^*(G_s(\Lambda) \times \mathbb{Z}^k)$.

Theorem:
$R_s(\Lambda)$ is strongly Morita equivalent to $C^*(\Lambda)$. Hence, $R_s(\Lambda)$ is nuclear and in the bootstrap class for which the UCT holds. Moreover, if Λ satisfies the aperiodicity condition, then $R_s(\Lambda)$ is simple, stable and purely infinite.

Hence, by the Kirchberg-Phillips theorem the isomorphism class of $R_s(\Lambda)$ is determined by its K-theory.

Λ satisfies the aperiodicity condition if there is a path $x \in \Lambda^{\Omega}$ which is not eventually periodic.

Idea of proof:

Use the notion of equivalence of groupoids of Muhly, Renault and Williams to show that the associated C*-algebras are strongly Morita equivalent.

Note that $C^*(\Lambda) = C^*(\mathcal{G}_\Lambda)$ where $\mathcal{G}_\Lambda$ is the path groupoid of Λ; the unit space is identified with Λ^Ω, the one-sided path space. The elements of $\mathcal{G}_\Lambda$ are triples (x, n, y) with $x, y \in \Lambda^\Omega$, $n \in \mathbb{Z}^k$ such that $\sigma^\ell x = \sigma^m y$ and $n = \ell - m$ for some $\ell, m \in \mathbb{N}^k$.

The equivalence of groupoids follows from the isomorphism:

$$G_s(\Lambda) \times \mathbb{Z}^k \cong \Lambda^\Delta \star \mathcal{G}_\Lambda \star \Lambda^\Delta$$

given by

$$((x, y), n) \mapsto (x, (\pi(x), n, \pi(\sigma^n y)), \sigma^n y),$$

where $\pi : \Lambda^\Delta \to \Lambda^\Omega$ is the restriction map.

Selected references

Cuntz and Krieger, *A class of C*-algebras and topological Markov chains,* 1980

Kaminker, Putnam and Spielberg, *Operator algebras and hyperbolic dynamics*, 1997

Kumjian and Pask, *Higher rank graph C*-algebras*, 2000

Muhly, Renault and Williams, *Equivalence and isomorphism for groupoid C*-algebras,* 1987

Putnam, *C*-algebras from Smale spaces*, 1996

Robertson and Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras,* 1999

Ruelle, *Thermodynamic Formalism*, 1978