Conjugacy of Functional Transducers

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Automata Theory and Symbolic Dynamics Workshop

Functional Transducers

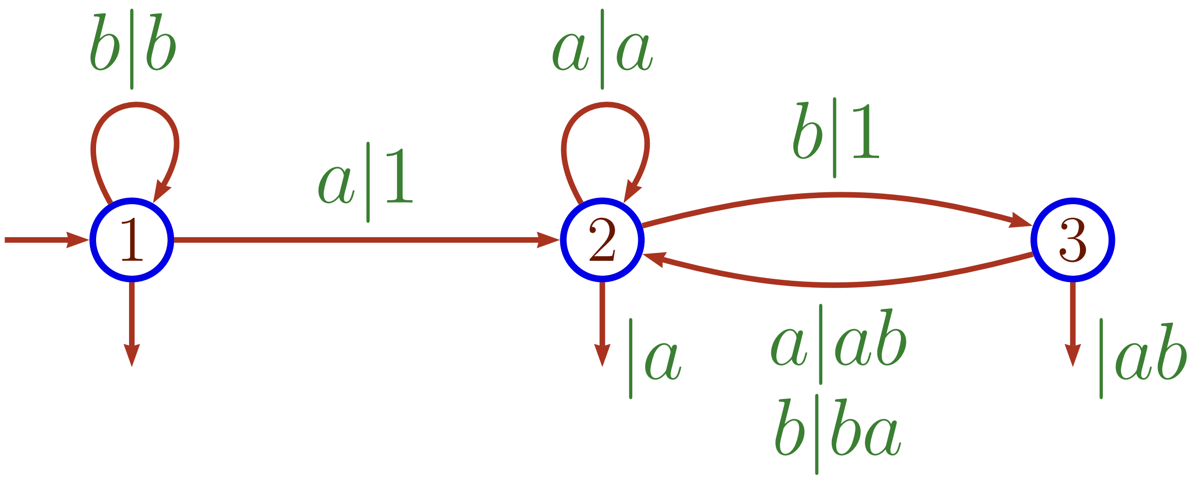
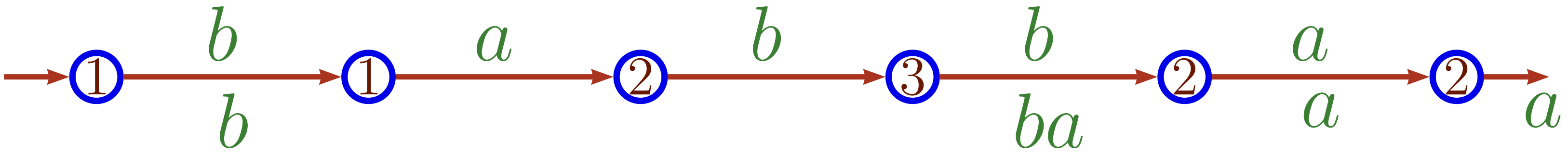


Image of $babba$



Functional Transducers

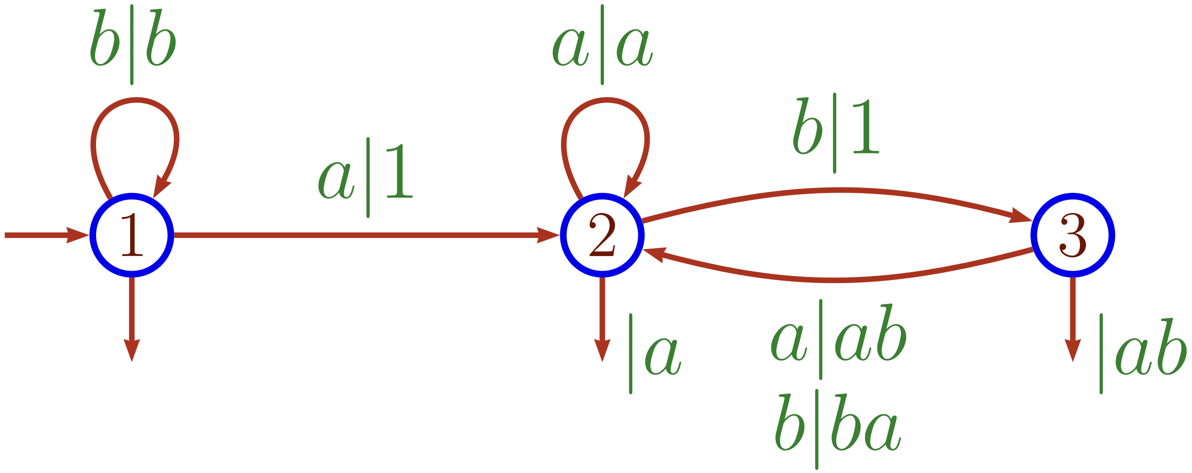


Image of $babba$: $bbaaa$

This transducer realizes a (rational) function from A^* into B^*

Functional Transducers

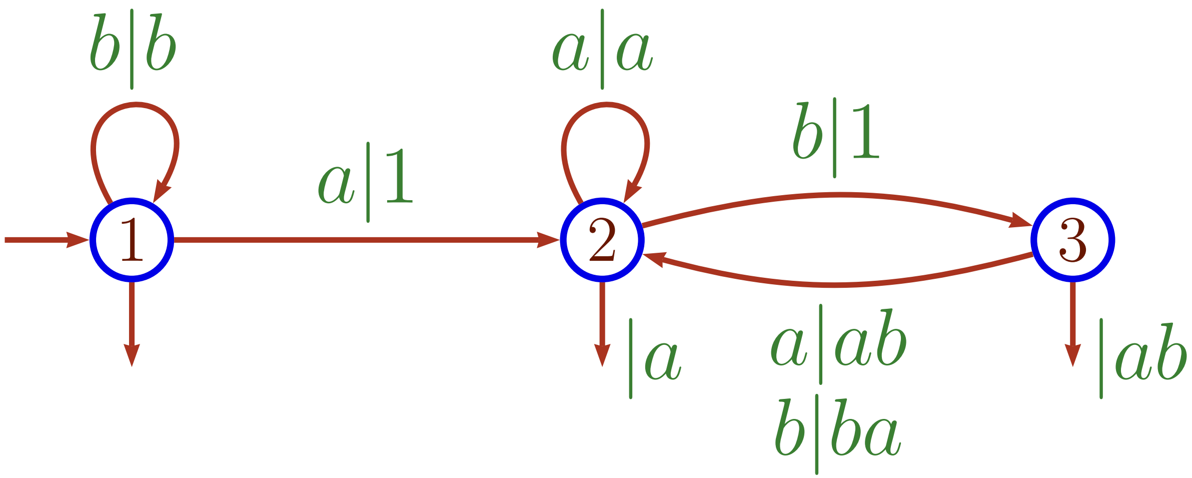
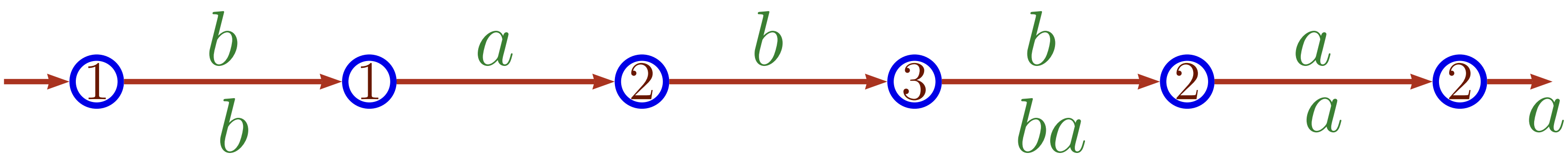


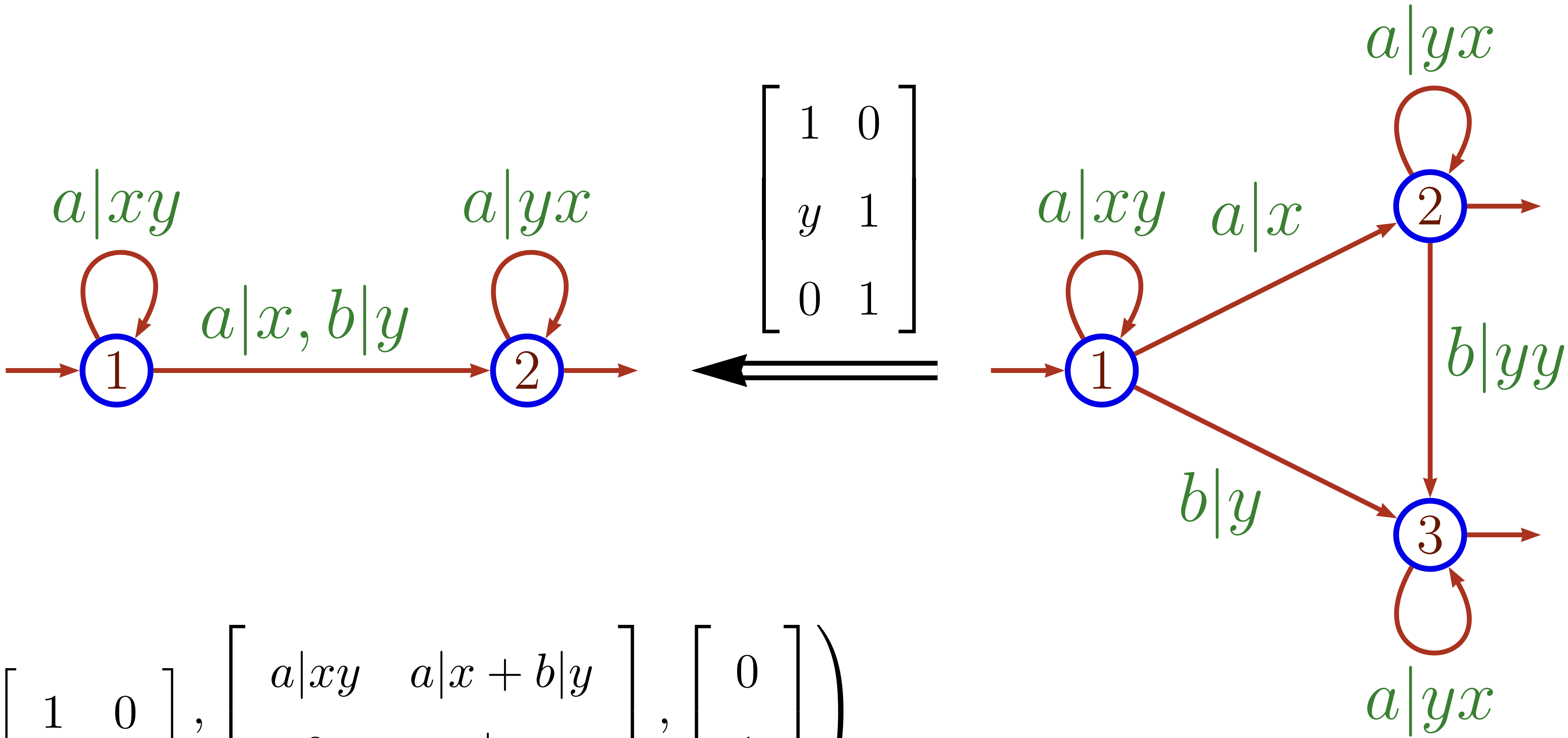
Image of $babba$: $bbaaa$



This transducer is sequential (input deterministic)

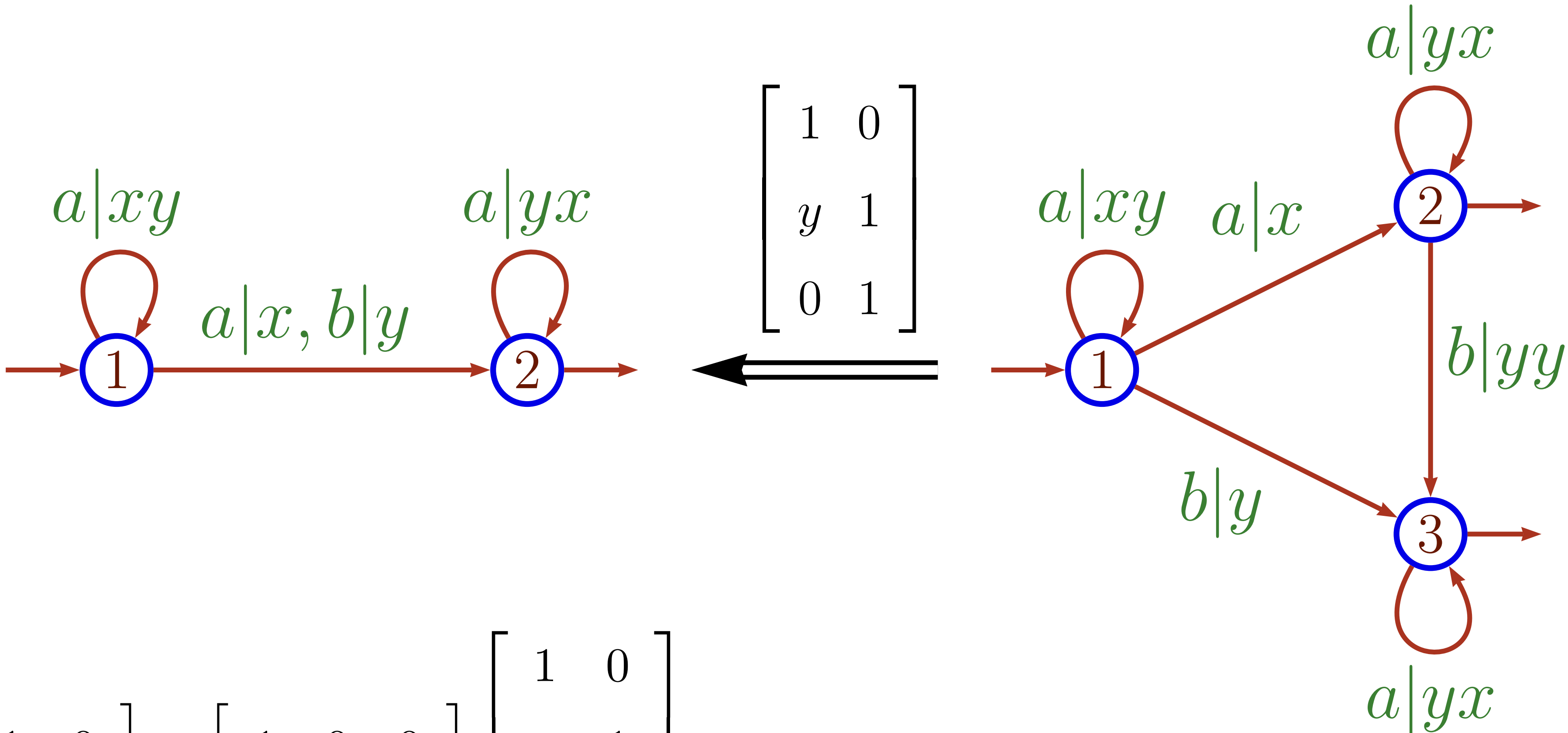
Matrix representation and conjugacy

Conjugacy



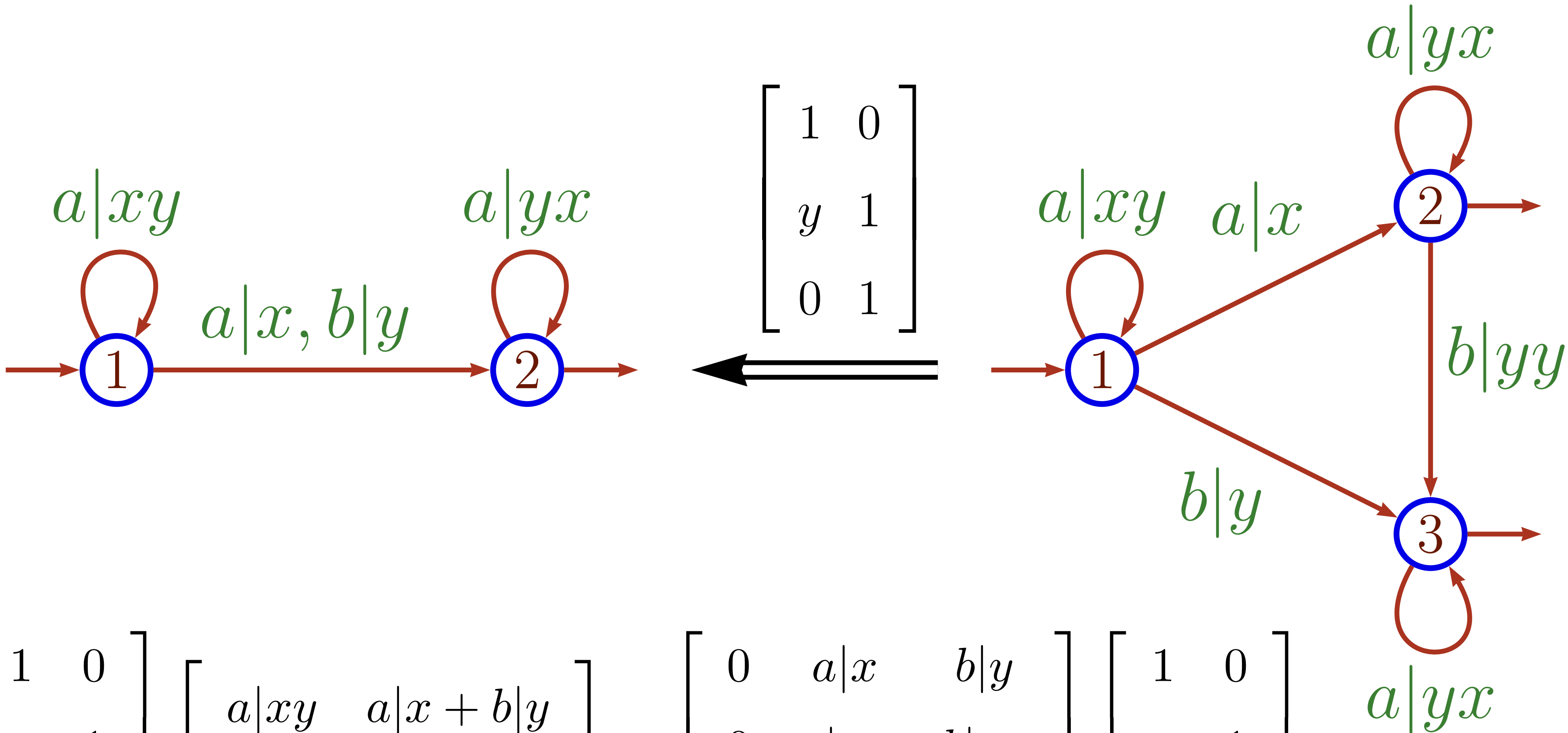
$$\left(\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} a|xy & a|x + b|y \\ 0 & a|yx \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Conjugacy



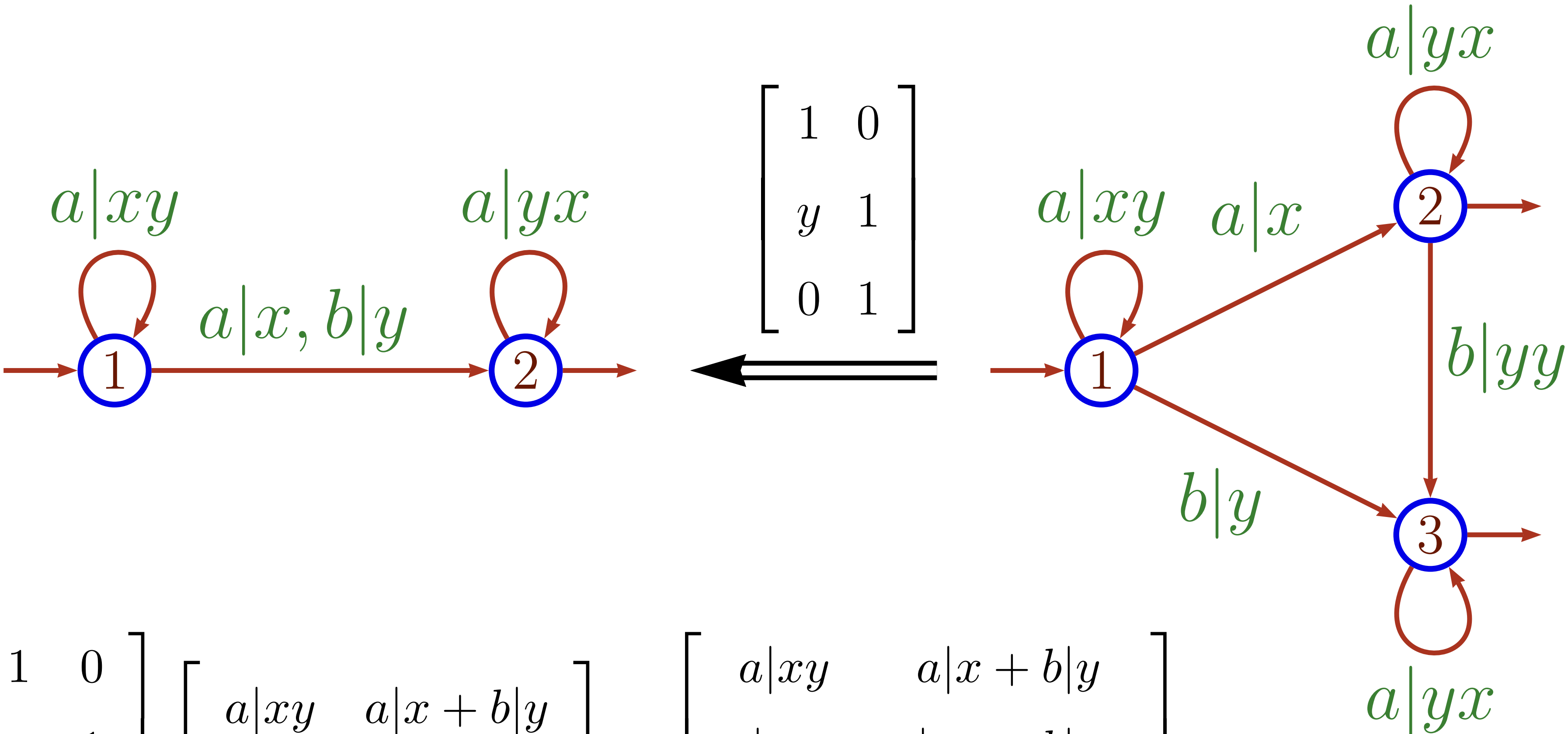
$$\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \\ 0 & 1 \end{bmatrix}$$

Conjugacy



$$\begin{bmatrix}1&0\\y&1\\0&1\end{bmatrix}\begin{bmatrix}a|xy & a|x+b|y\\0&a|yx\end{bmatrix}=\begin{bmatrix}0&a|x&b|y\\0&a|yx&b|yy\\0&0&a|yx\end{bmatrix}\begin{bmatrix}1&0\\y&1\\0&1\end{bmatrix}$$

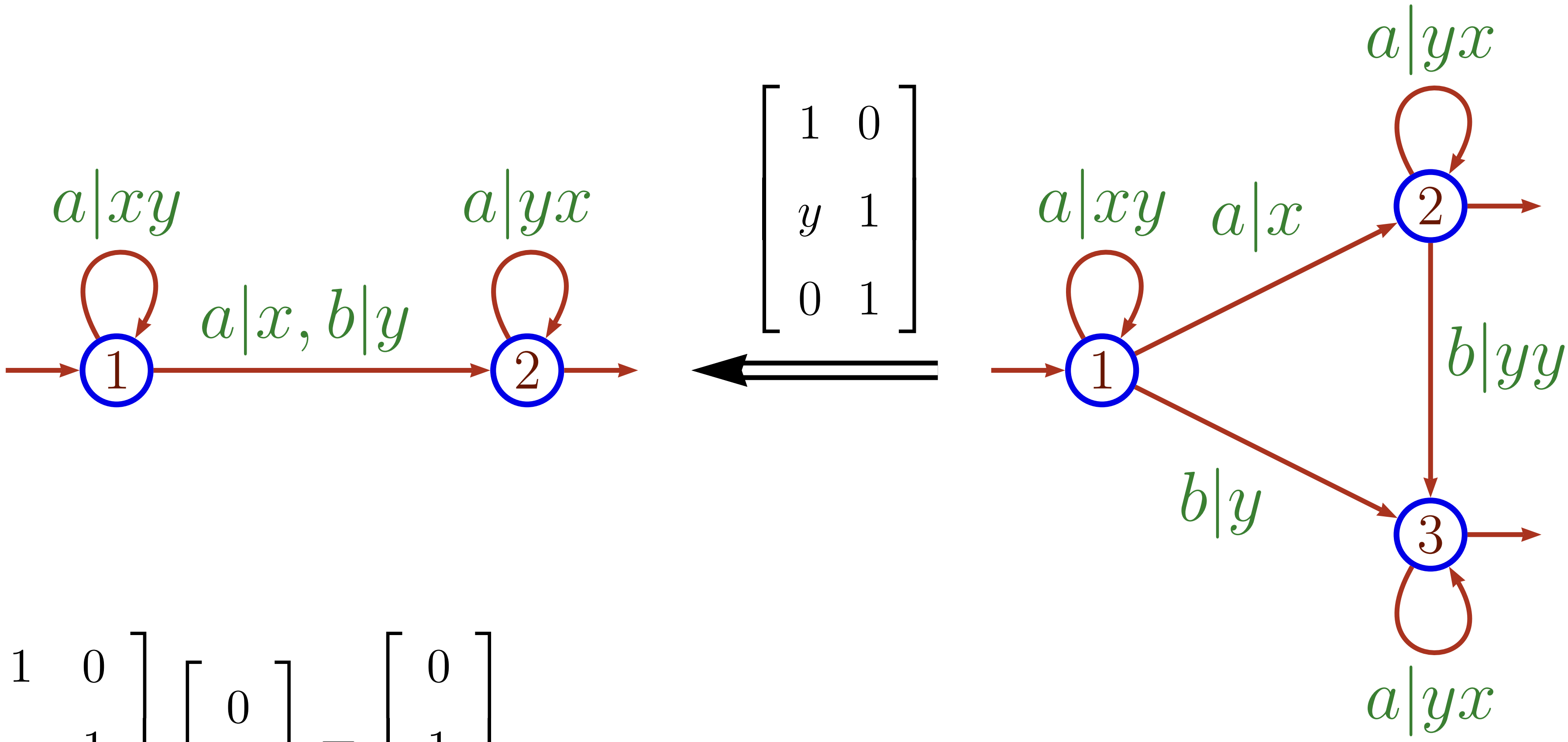
Conjugacy



$$\begin{bmatrix} 1 & 0 \\ y & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a|xy & a|x + b|y \\ 0 & a|yx \end{bmatrix} = \begin{bmatrix} a|xy & a|x + b|y \\ a|yxy & a|yx + b|yy \\ 0 & a|yx \end{bmatrix}$$

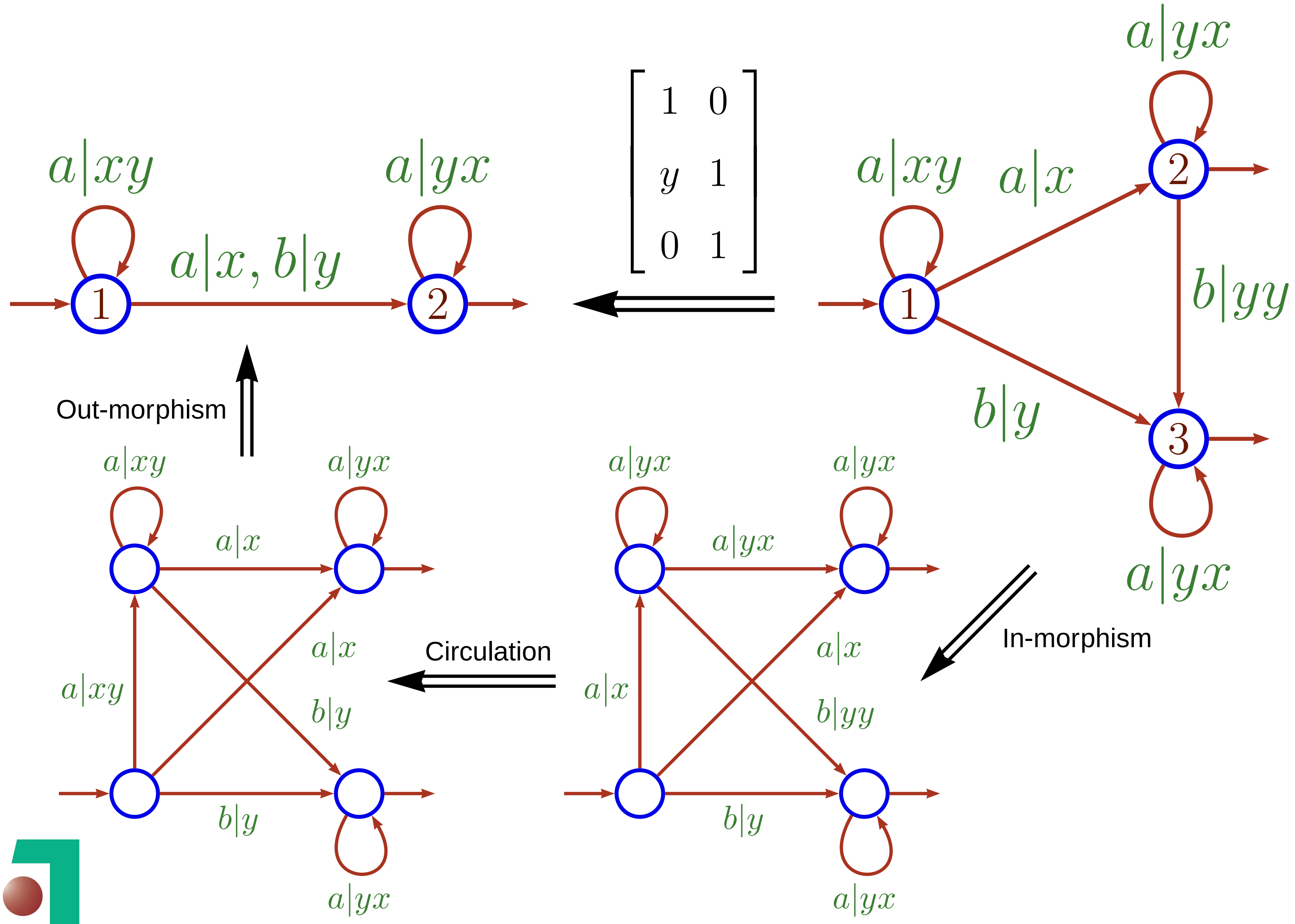
The semiring of multiplicities is $\mathcal{P}_f(B^*)$

Conjugacy



$$\begin{bmatrix} 1 & 0 \\ y & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Conjugacy


a|xy
a|yx
a|x, b|y
1
2
1 0
y 1
0 1
a|yx
a|xy
a|x
b|yy
b|y
a|yx
1
2
3
Out-morphism
In-morphism
Circulation
a|xy
a|yx
a|x
a|xy
a|x
b|y
b|y
a|yx
a|yx
a|yx
a|yx
a|x
a|x
b|yy
b|y
a|yx

Conjugacy theorem

Theorem.

$\mathcal{A}_1$ and $\mathcal{A}_2$ functional transducers

$$\mathcal{A}_1 \equiv \mathcal{A}_2 \implies \exists \mathcal{B},\ \mathcal{A}_1 \overset{X_1}{\Longleftarrow} \mathcal{B} \overset{X_2}{\Longrightarrow} \mathcal{A}_2$$

Sequentialisation and Pseudo-sequentialisation

Functional Transducers

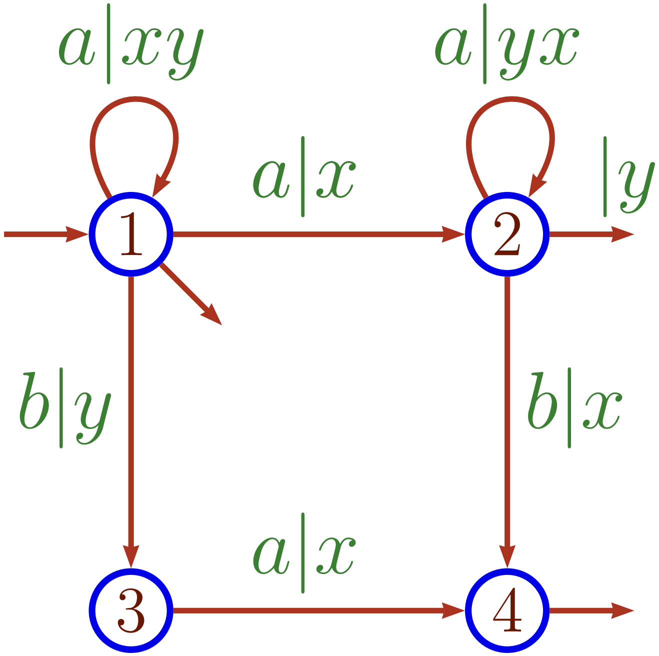
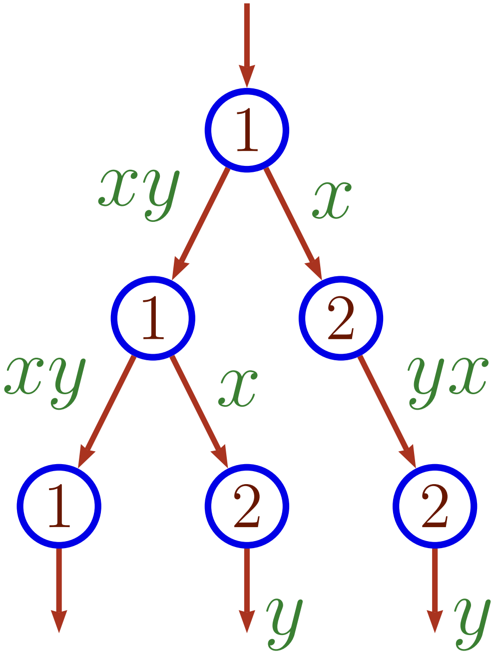


Image of aa :



Functional Transducers

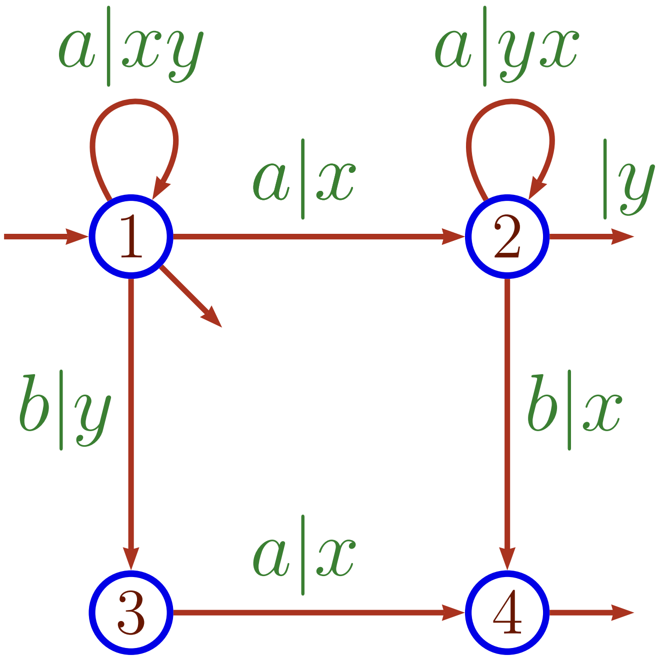
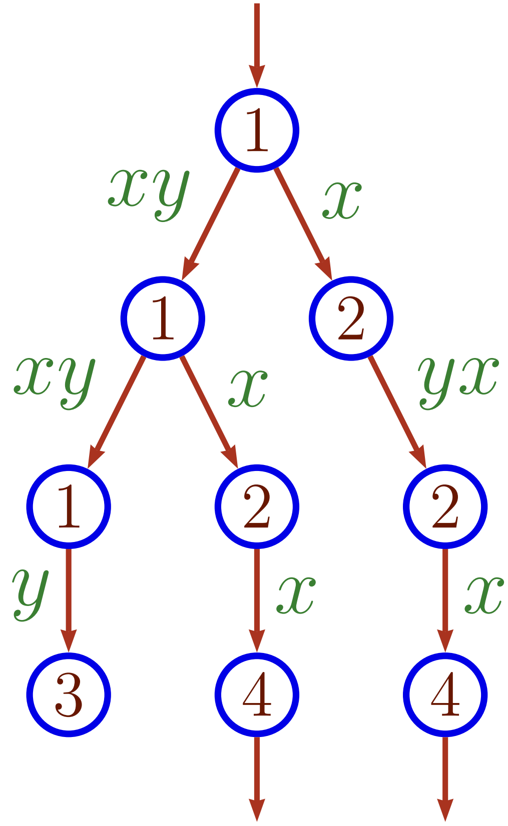


Image of aab:



Functional Transducers

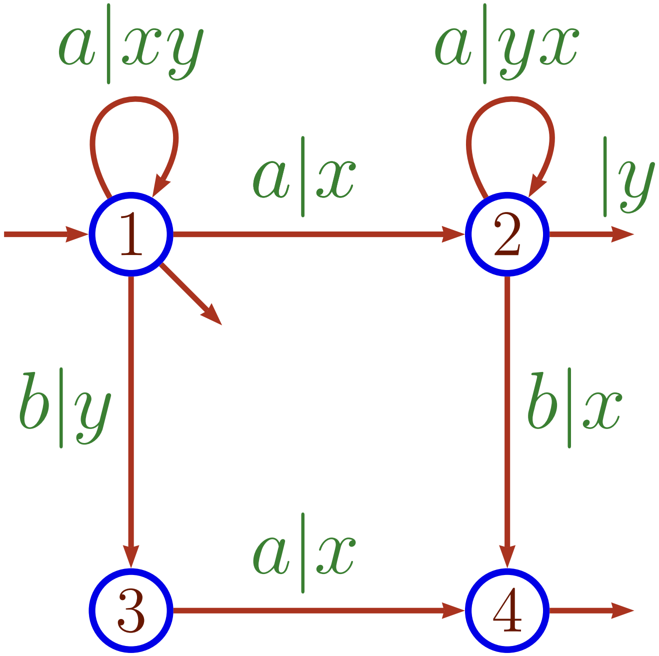
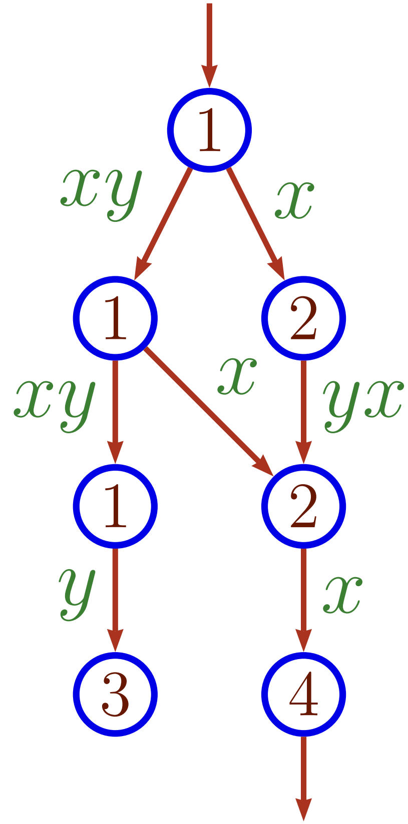


Image of aab:



Functional Transducers

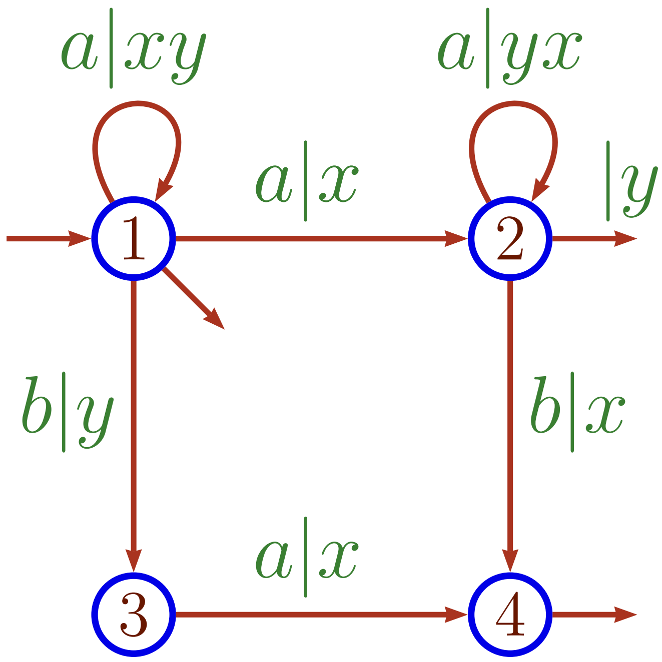
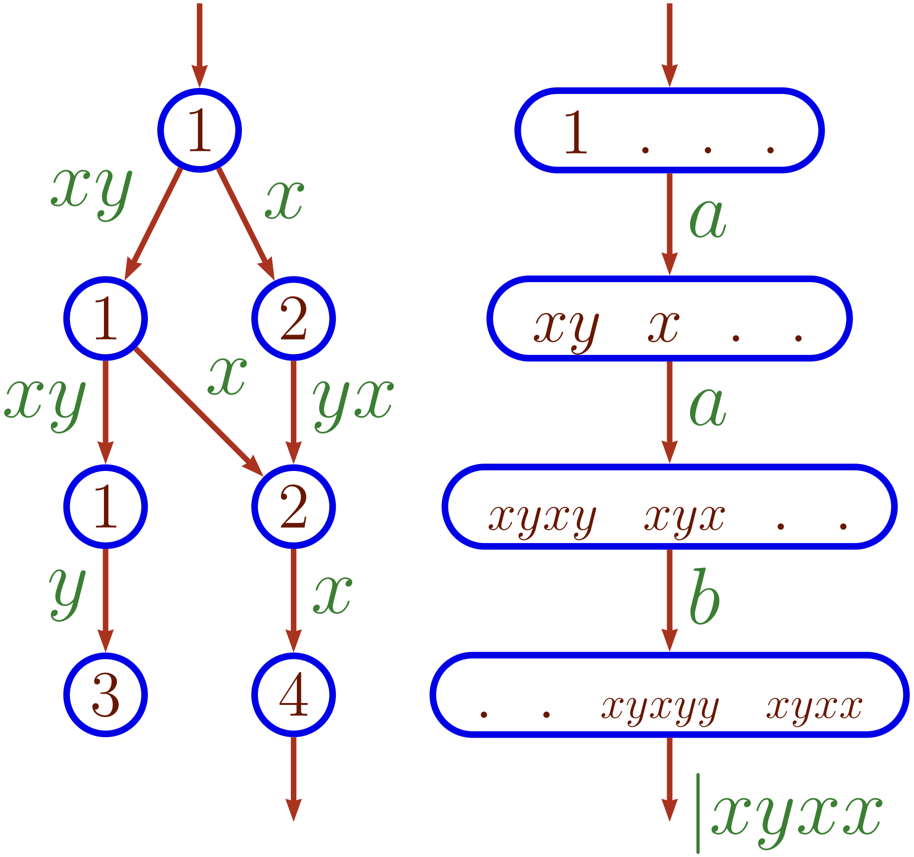


Image of aab:



Functional Transducers

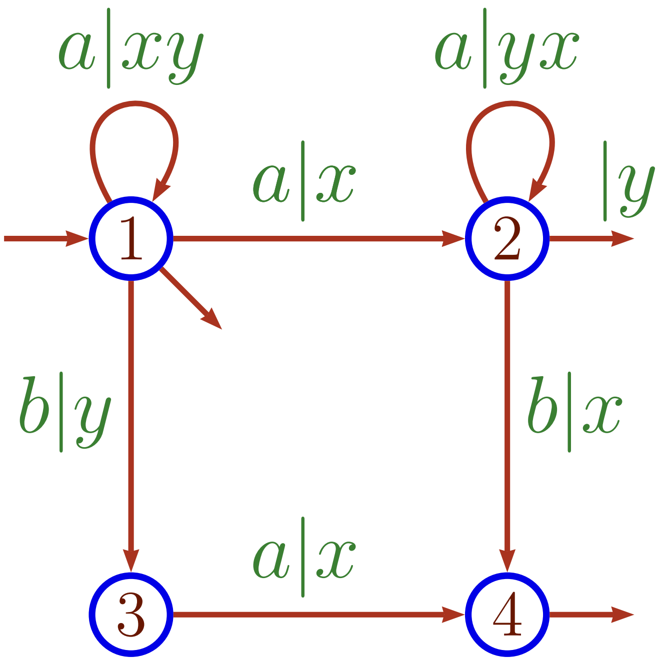
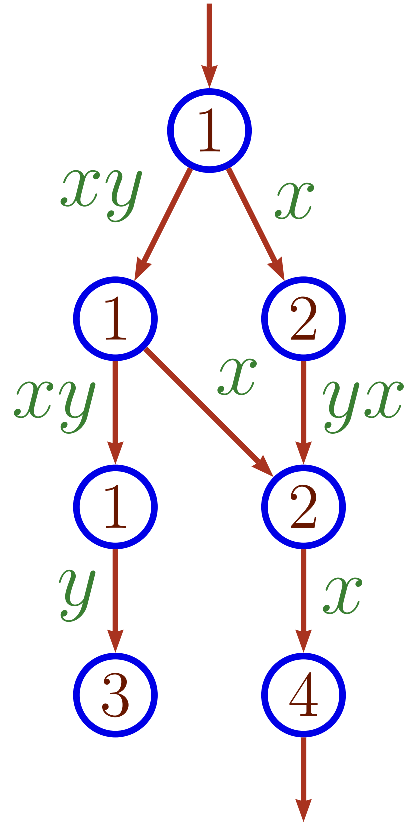
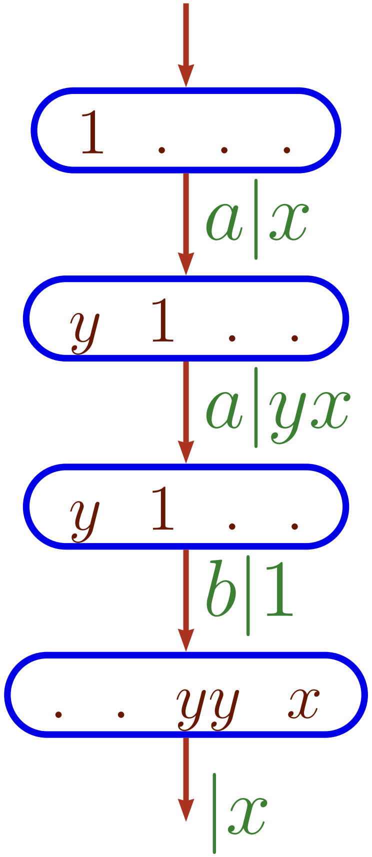
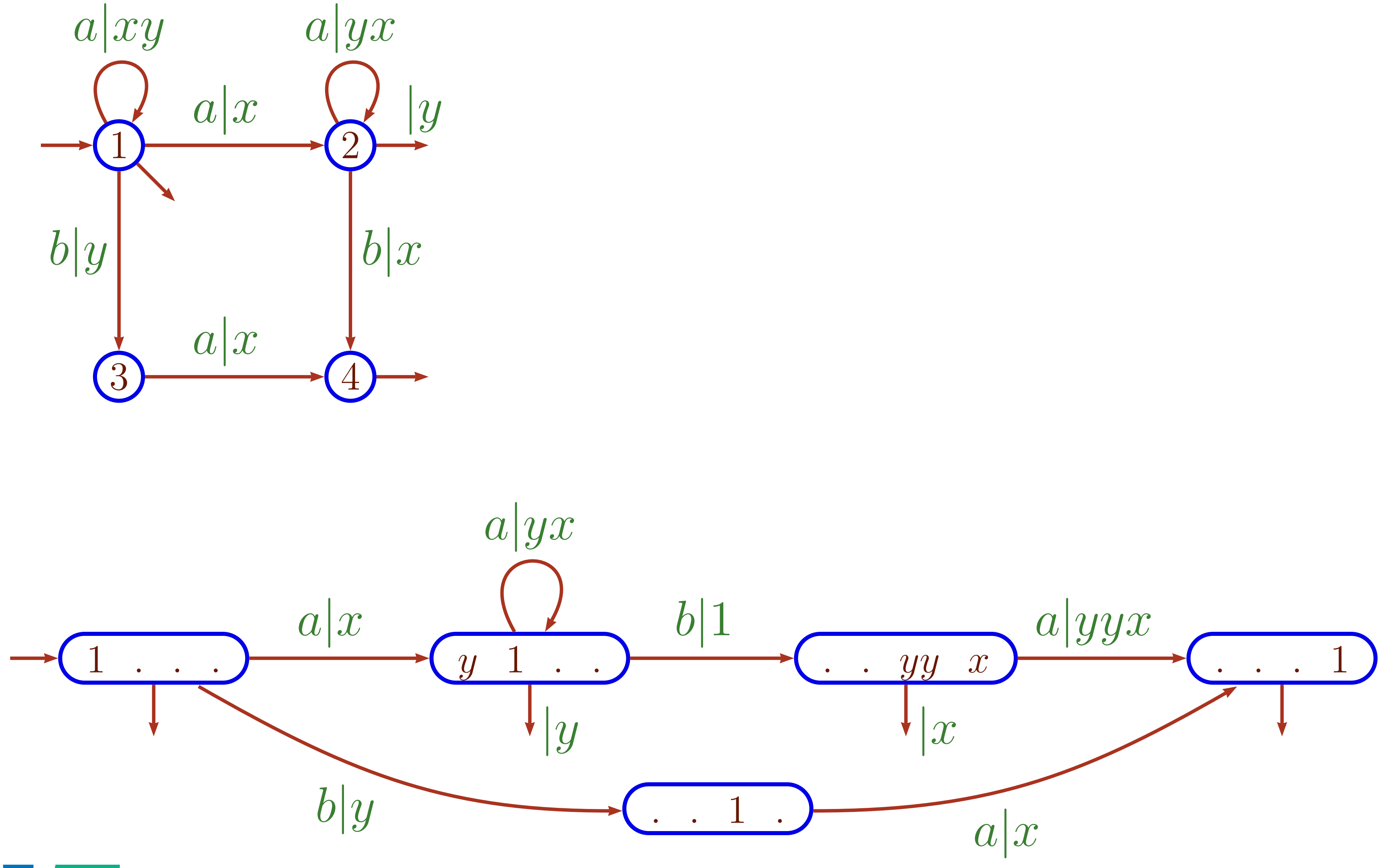


Image of aab:

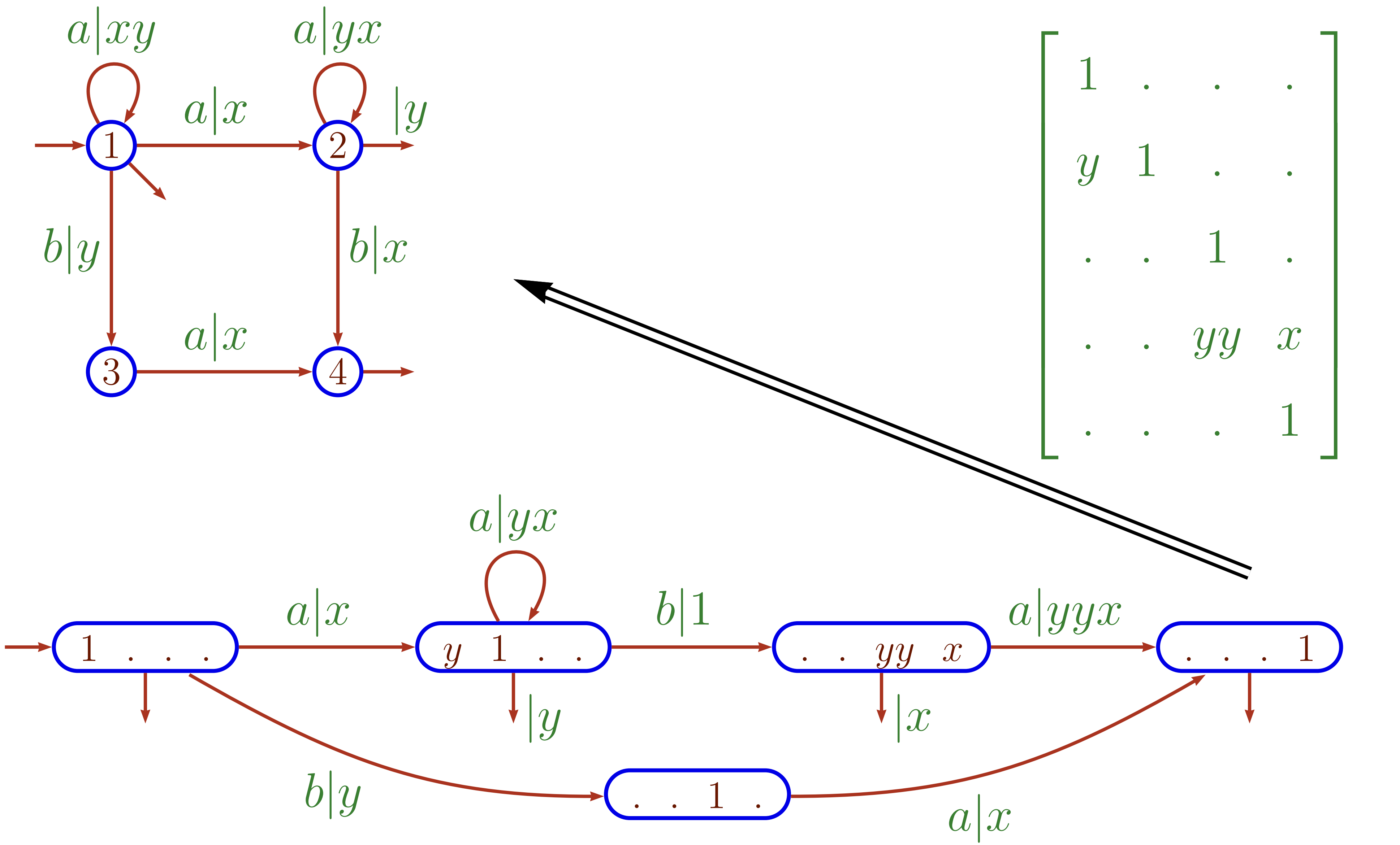




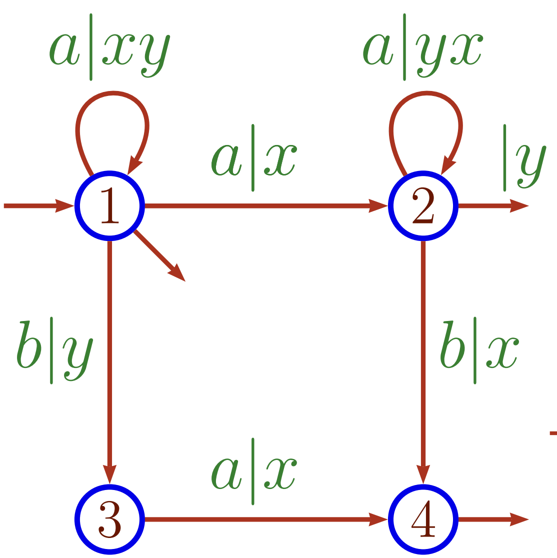
Sequentialisation

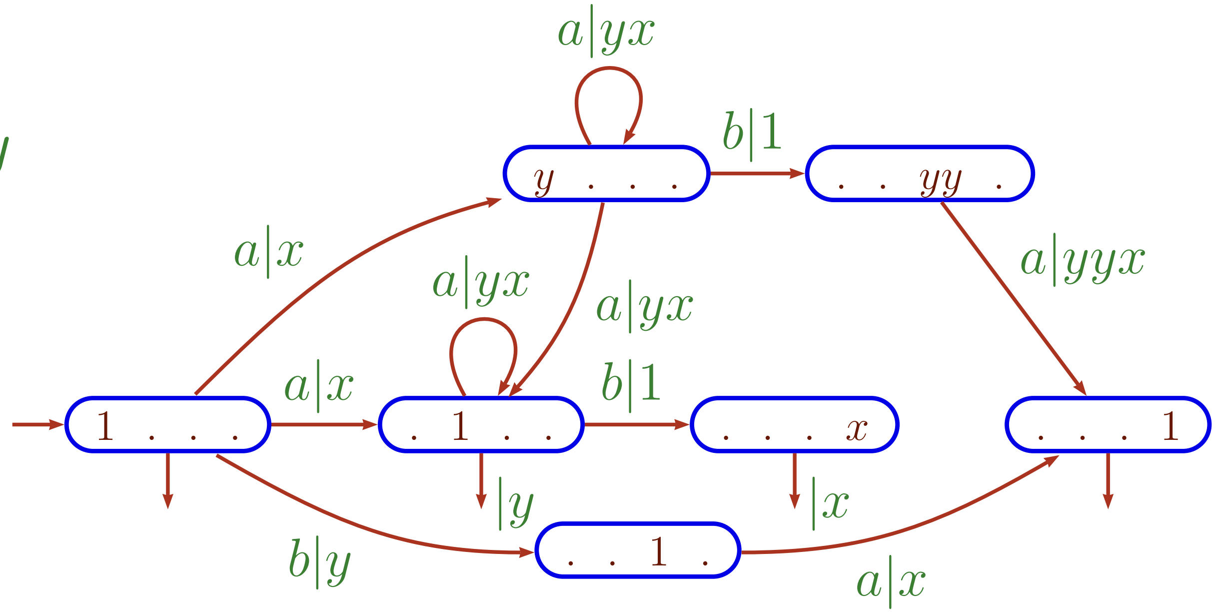


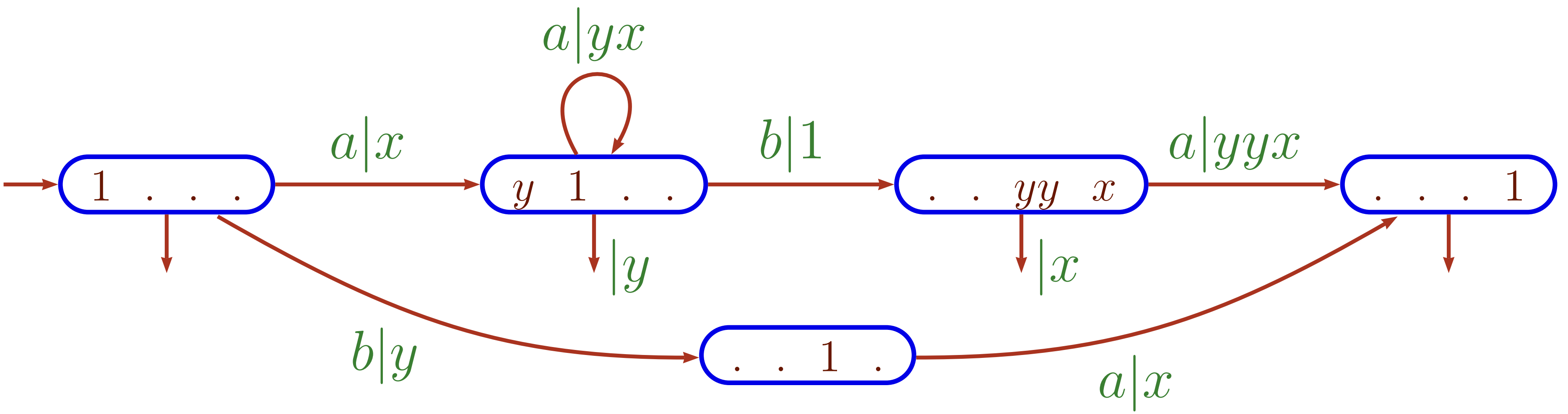
Sequentialisation



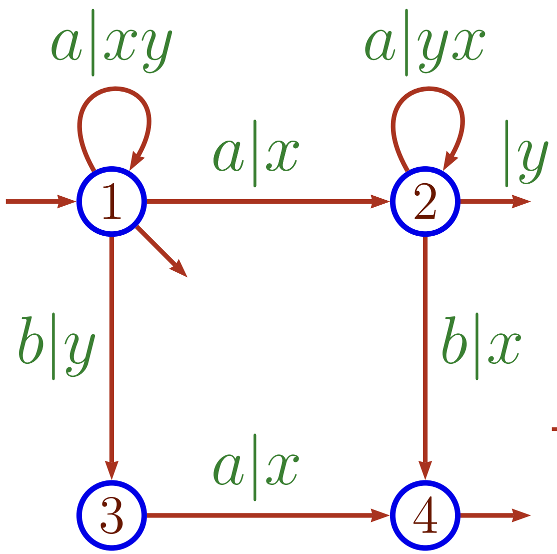
Sequentialisation

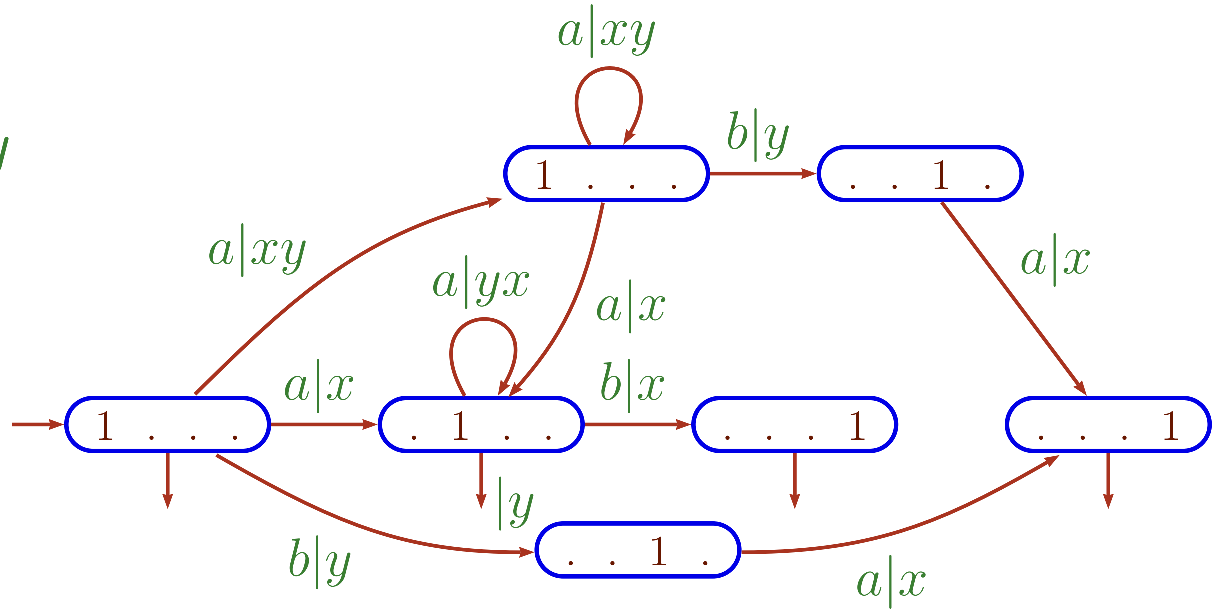


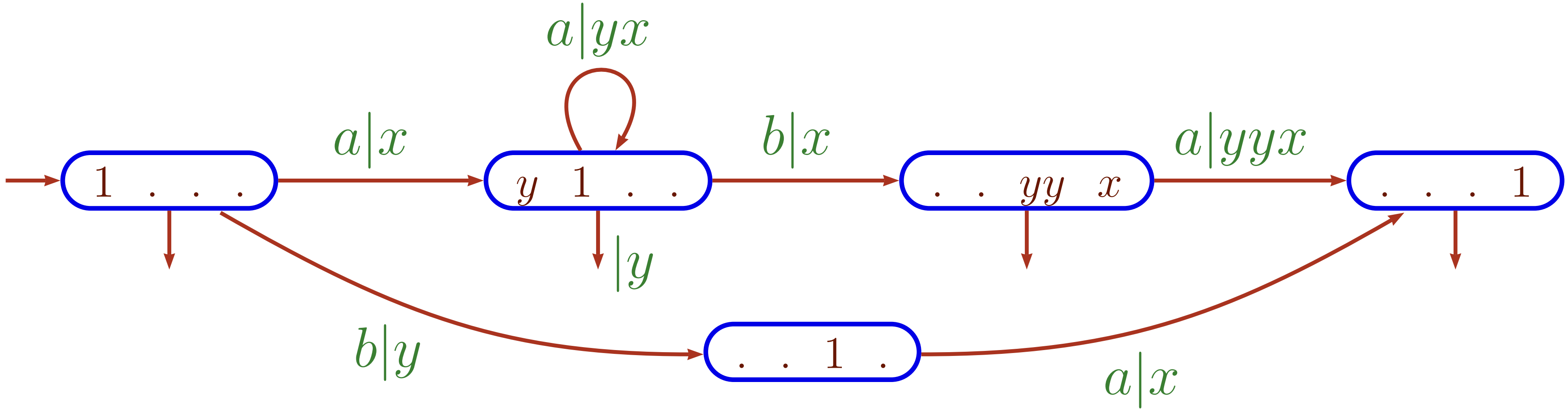




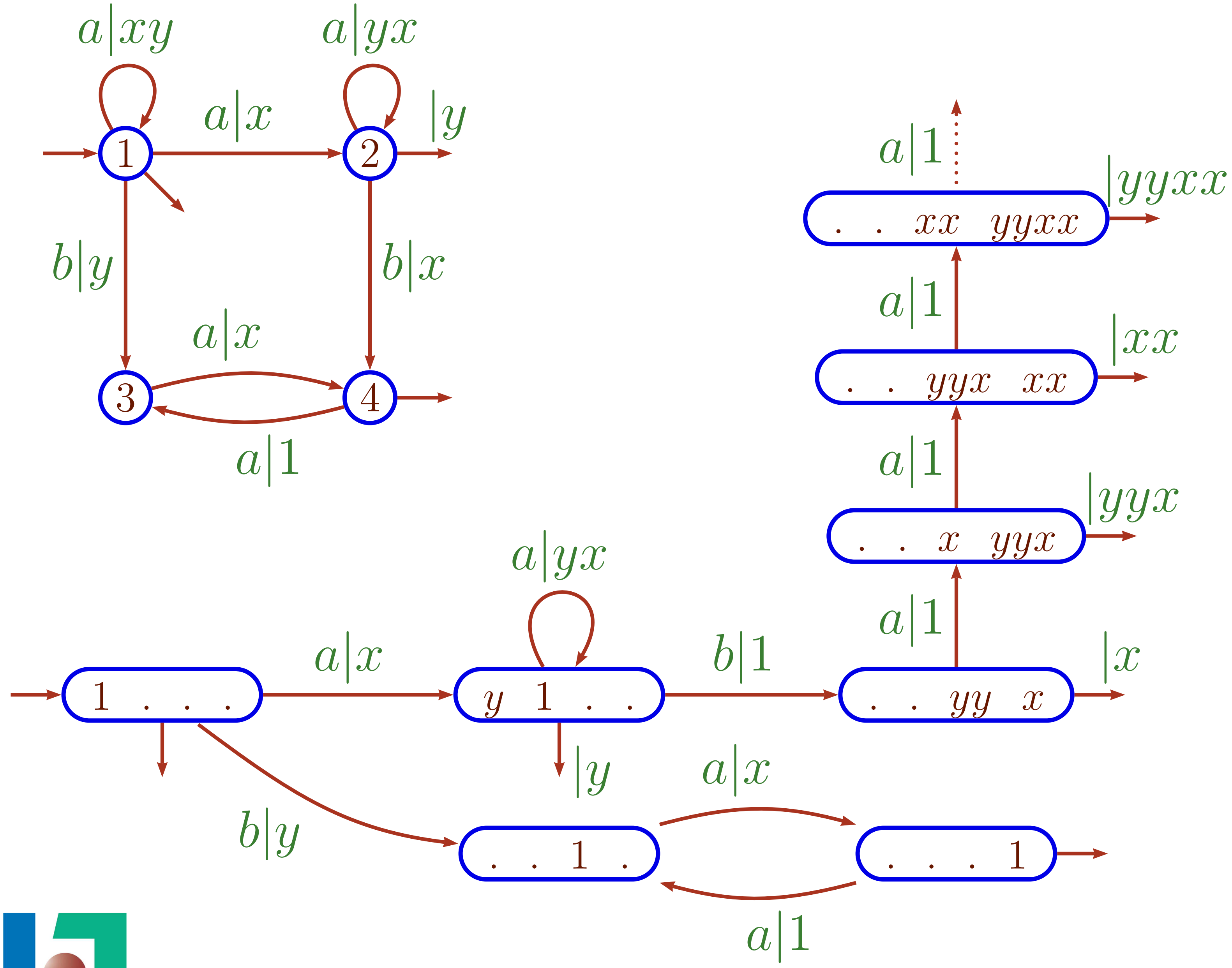
Sequentialisation



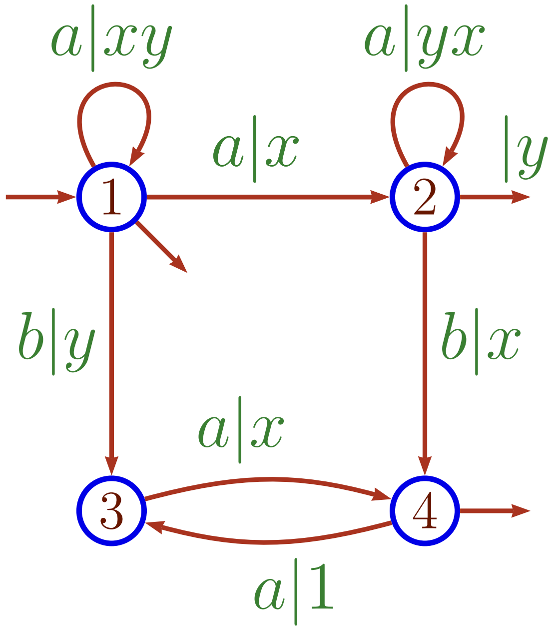


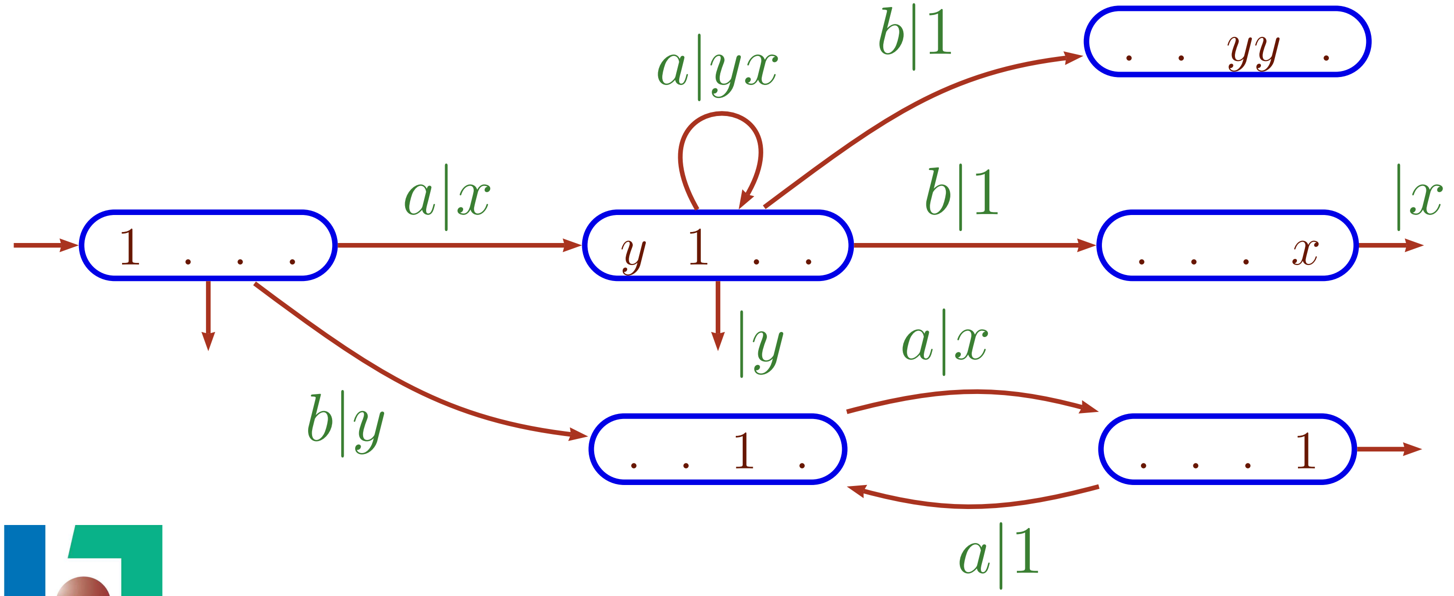


Non Sequential Function

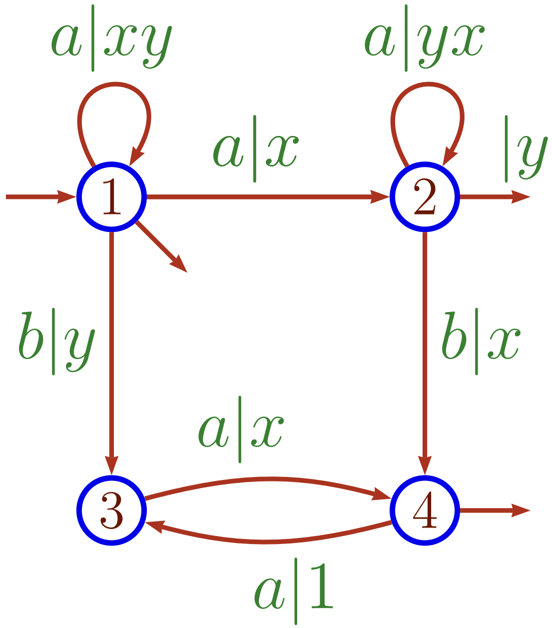


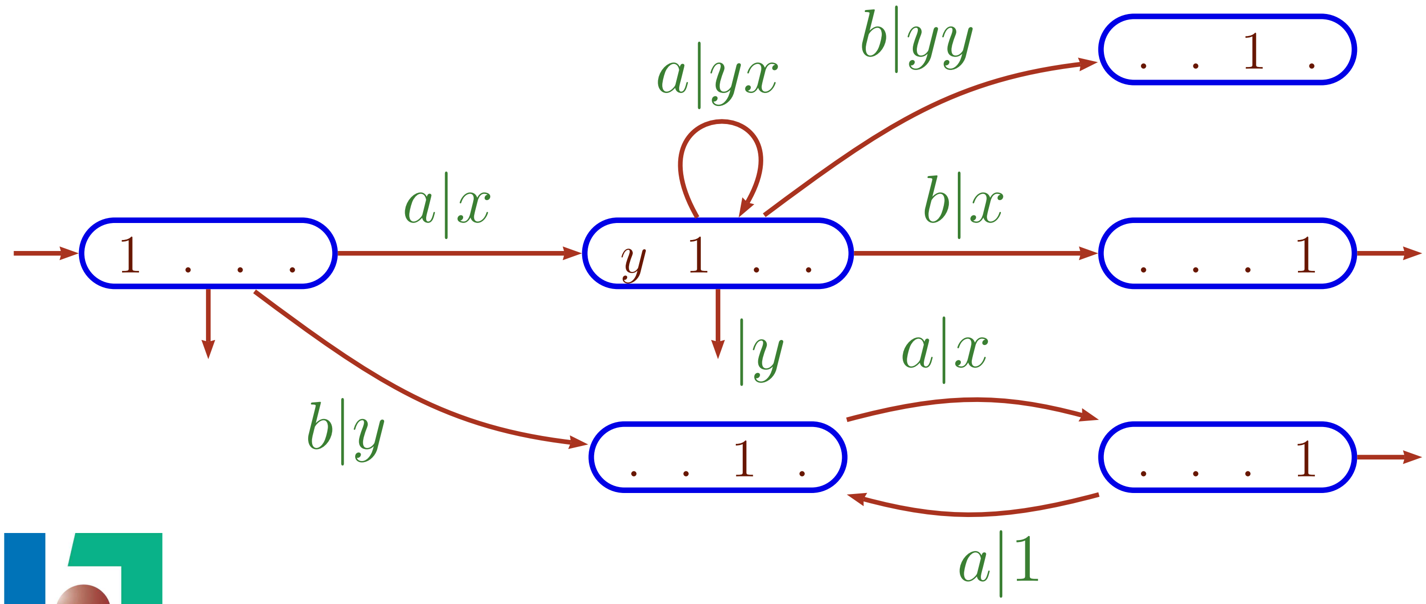
Non Sequential Function



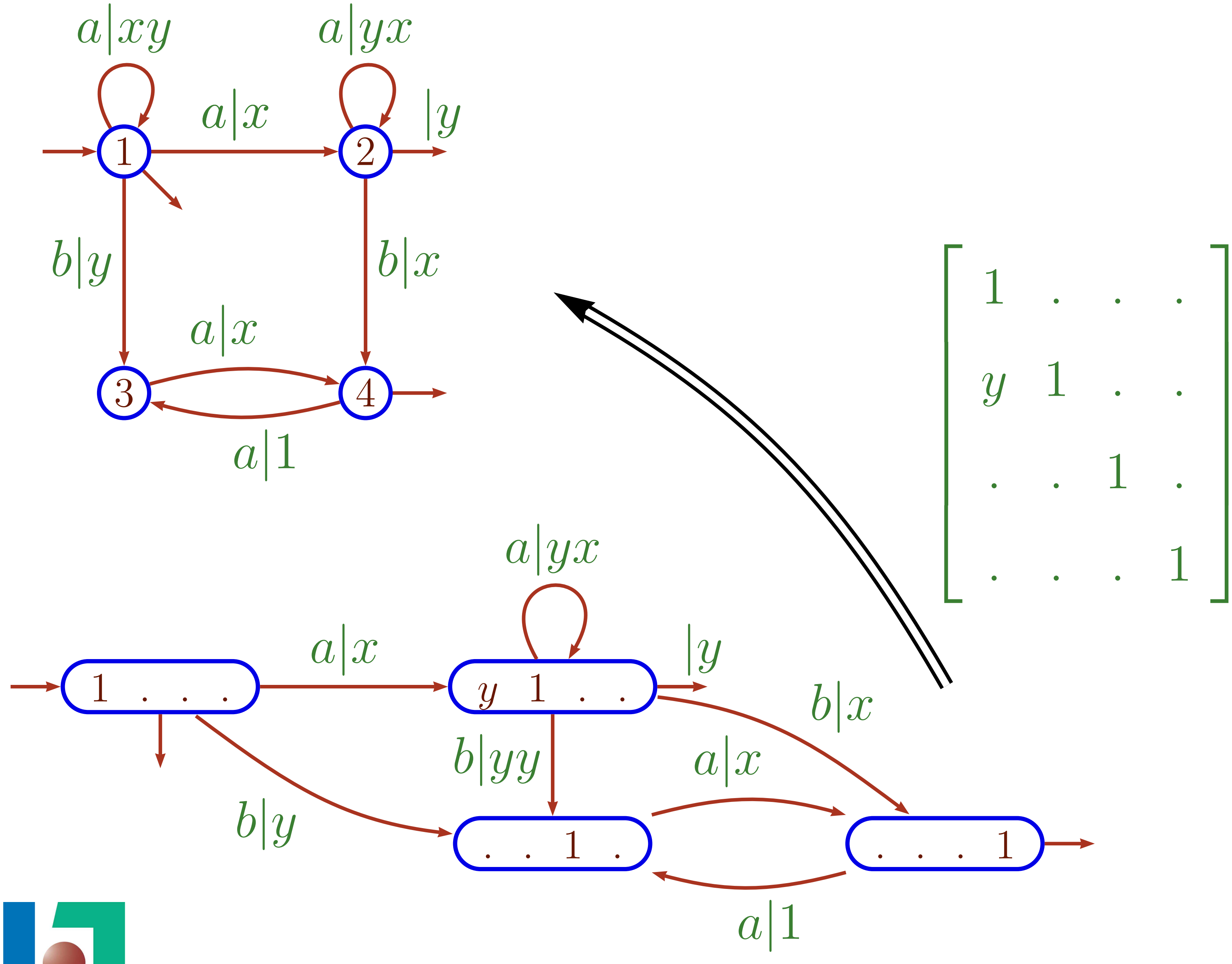


Non Sequential Function

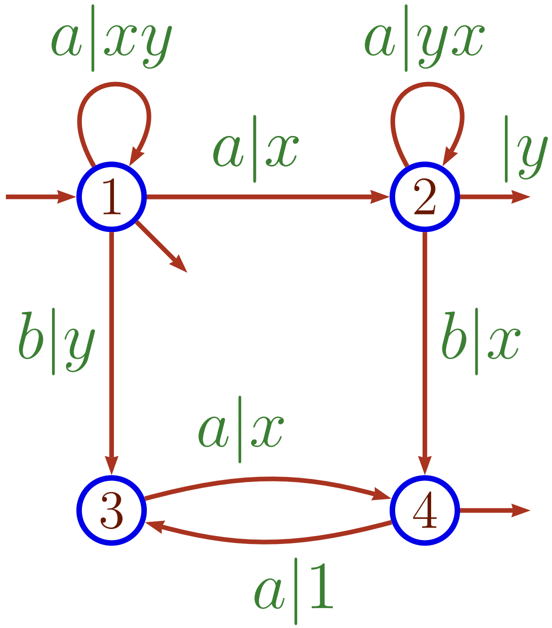




Non Sequential Function

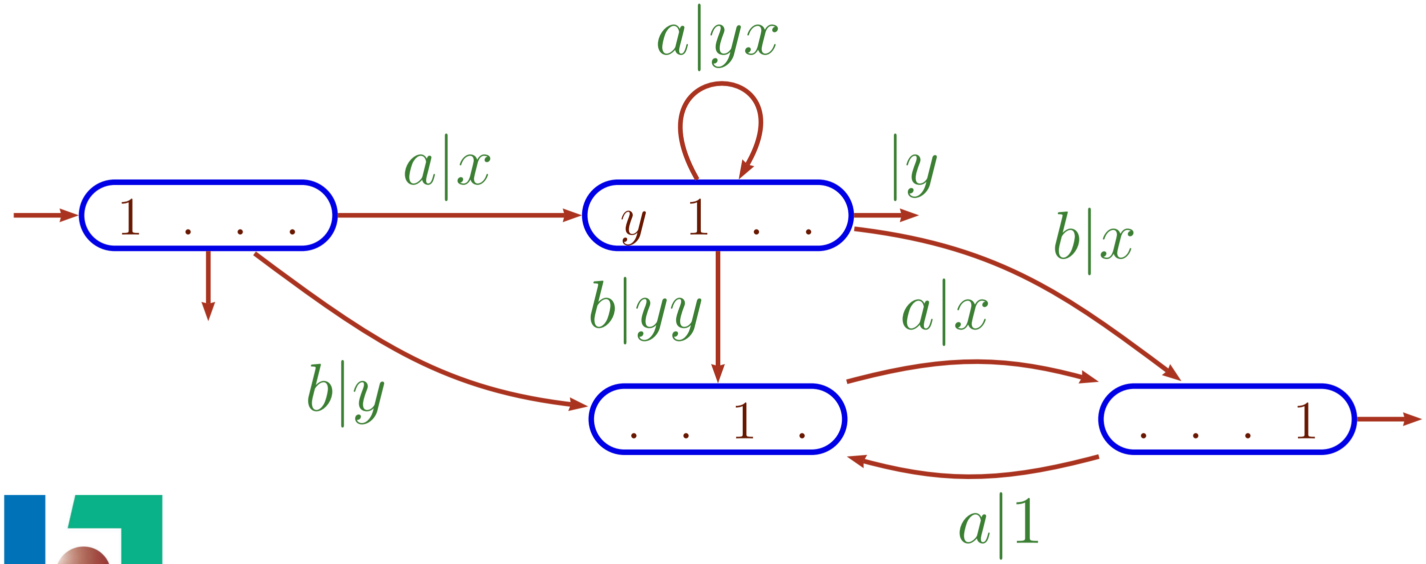


Non Sequential Function



This automaton is unambiguous:

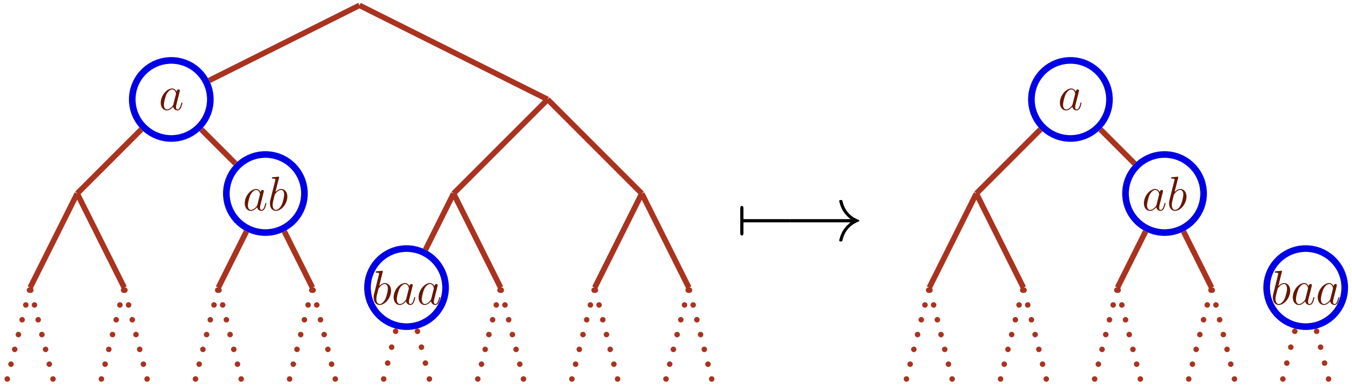
(At most one computation for each word)



Pseudo-sequentialisation

Criteria for splitting states:

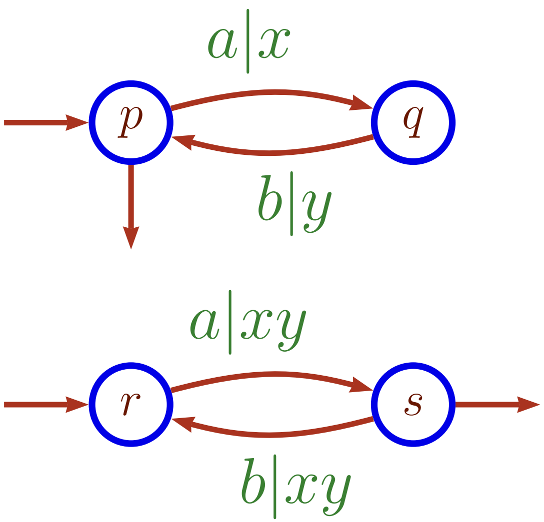
- Every simplified state must contain 1;

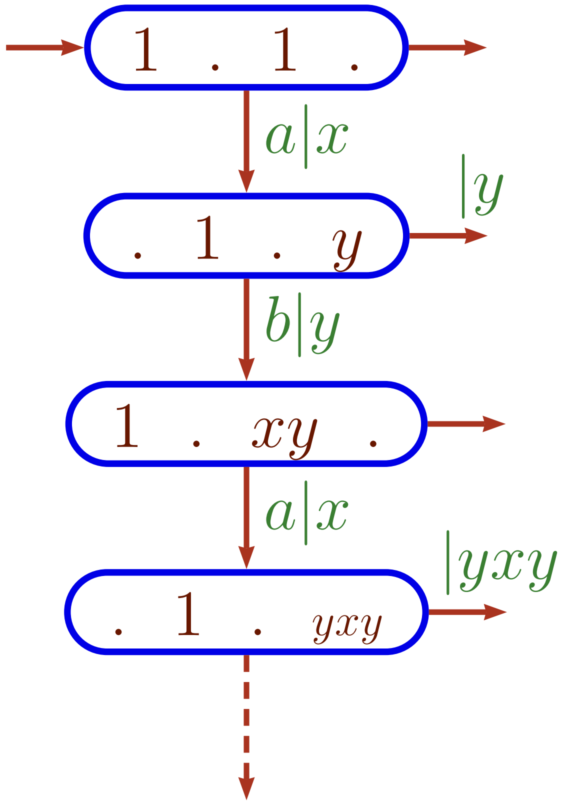


Pseudo-sequentialisation

Criteria for splitting states:

- Every simplified state must contain 1;
- The maximal gap in the set of entrees must be smaller than a constant K.





Pseudo-sequentialisation

Criteria for splitting states:

- Every simplified state must contain 1;
- The maximal gap in the set of entrees must be smaller than a constant K.

Proposition. Let $\mathcal{A}$ with n states. Let M be the maximal length of outputs of transitions.
If $K > n^2 M$, the pseudo-sequentialisation of $\mathcal{A}$ is unambiguous.

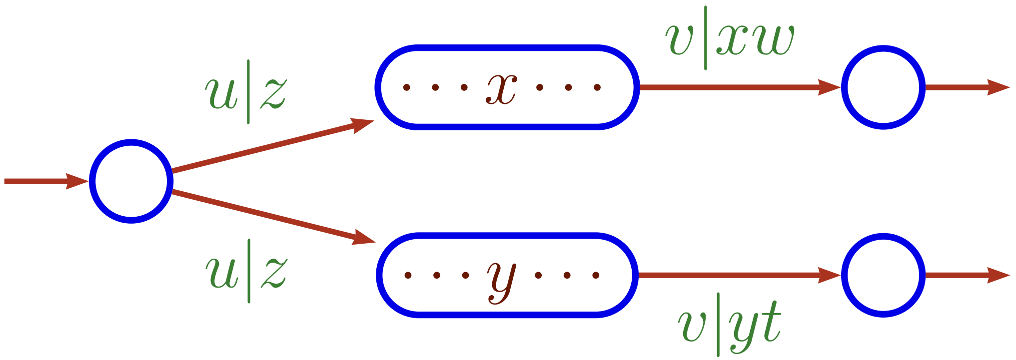
Idea of the proof

We show that from two states coming from a split, one can not accept the same word.

Assume that there exists such a word v.

First case:

The words x and y are uncompatible:



$$|xw| \neq |yt|$$

Idea of the proof

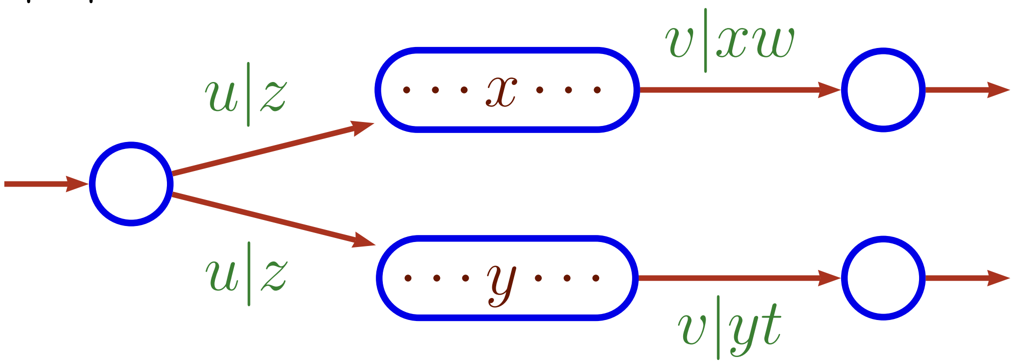
We show that from two states coming from a split, one can not accept the same word.

Assume that there exists such a word v.

Second case:

There is a large gap between x and y: $|x| - |y| > Mn^2$

We can assume $|v| \leqslant n^2 - 1$



$$|xw| - |yt| > Mn^2 + |w| - |t| > 0$$

Joint pseudo-sequentialisation

Joint pseudo-sequentialisation

If $\mathcal{A}_1 = (I_1, E_1, T_1)$ and $\mathcal{A}_2 = (I_2, E_2, T_2)$ are equivalent, so is

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \left(\left[\begin{array}{c|c} I_1 & I_2 \end{array}\right], \left[\begin{array}{c|c} E_1 & . \\ \hline . & E_2 \end{array}\right], \left[\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right] \right)$$

Let $\mathcal{B} = (J, F, U)$ be the pseudo-sequentialisation of this union :

$$\mathcal{A}_1 \cup \mathcal{A}_2 \xLeftarrow{[X_1|X_2]} \mathcal{B}$$

We want to show

$$\begin{cases} \mathcal{A}_1 \xLeftarrow{X_1} \mathcal{B} \\ \mathcal{A}_2 \xLeftarrow{X_2} \mathcal{B} \end{cases}$$

Joint pseudo-sequentialisation

If $\mathcal{A}_1 = (I_1, E_1, T_1)$ and $\mathcal{A}_2 = (I_2, E_2, T_2)$ are equivalent, so is

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \left(\left[\begin{array}{c|c} I_1 & I_2 \end{array}\right], \left[\begin{array}{c|c} E_1 & \cdot \\ \hline \cdot & E_2 \end{array}\right], \left[\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right] \right)$$

Let $\mathcal{B} = (J, F, U)$ be the pseudo-sequentialisation of this union :

$$\mathcal{A}_1 \cup \mathcal{A}_2 \xLeftarrow{[X_1|X_2]} \mathcal{B}$$

$$J \cdot \left[\begin{array}{c|c} X_1 & X_2 \end{array}\right] = \left[\begin{array}{c|c} I_1 & I_2 \end{array}\right] \Longrightarrow \begin{cases} J \cdot X_1 = I_1 \\ J \cdot X_2 = I_2 \end{cases}$$

Joint pseudo-sequentialisation

If $\mathcal{A}_1 = (I_1, E_1, T_1)$ and $\mathcal{A}_2 = (I_2, E_2, T_2)$ are equivalent, so is

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \left(\left[\begin{array}{c|c} I_1 & I_2 \end{array}\right], \left[\begin{array}{c|c} E_1 & \cdot \\ \hline \cdot & E_2 \end{array}\right], \left[\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right] \right)$$

Let $\mathcal{B} = (J, F, U)$ be the pseudo-sequentialisation of this union :

$$\mathcal{A}_1 \cup \mathcal{A}_2 \overset{[X_1|X_2]}{\Longleftarrow} \mathcal{B}$$

$$F \cdot \left[\begin{array}{c|c} X_1 & X_2 \end{array}\right] = \left[\begin{array}{c|c} X_1 & X_2 \end{array}\right] \cdot \left[\begin{array}{c|c} E_1 & \cdot \\ \hline \cdot & E_2 \end{array}\right]$$

$$\Longrightarrow \begin{cases} F \cdot X_1 = X_1 \cdot E_1 \\ F \cdot X_2 = X_2 \cdot E_2 \end{cases}$$

Joint pseudo-sequentialisation

If $\mathcal{A}_1 = (I_1, E_1, T_1)$ and $\mathcal{A}_2 = (I_2, E_2, T_2)$ are equivalent, so is

$$\mathcal{A}_1 \cup \mathcal{A}_2 = \left(\left[\begin{array}{c|c} I_1 & I_2 \end{array} \right], \left[\begin{array}{c|c} E_1 & \cdot \\ \hline \cdot & E_2 \end{array} \right], \left[\begin{array}{c} T_1 \\ \hline T_2 \end{array} \right] \right)$$

Let $\mathcal{B} = (J, F, U)$ be the pseudo-sequentialisation of this union :

$$\mathcal{A}_1 \cup \mathcal{A}_2 \overset{[X_1|X_2]}{\Longleftarrow} \mathcal{B}$$

$$U = \left[\begin{array}{c|c} X_1 & X_2 \end{array} \right] \cdot \left[\begin{array}{c} T_1 \\ \hline T_2 \end{array} \right] = X_1 \cdot T_1 \cup X_2 \cdot T_2$$

It implies $\begin{cases} U = X_1 \cdot T_1 \\ U = X_2 \cdot T_2 \end{cases}$ if $\mathcal{B}$ is unambiguous.

Remarks and conclusion

- The pseudo-sequentialisation can be tuned to give sequentialisation on sequentialisable transducers.
- There are (very) good reasons to think that nothing works for more general transducers.
- What is the status of $\mathbb{N}$-transducers ?
- What are the connections with symbolic dynamics ?