OPEN PROBLEMS SESSION 1

Notes by Karl Petersen

In addition to these problems, see Mike Boyle's Open Problems article [1].

1. (Brian Marcus)

Let X be a $\mathbb{Z}^d$ subshift over the alphabet $A = \{0, 1\}$. Let $B = \{0, 1, \square\}$ be a slightly larger alphabet, with a new symbol, $\square$, adjoined to A. For each $\hat{x} \in B^{\mathbb{Z}^d}$ let

$$\Phi(\hat{x}) = \{x \in A^{\mathbb{Z}^d} : x_v = \hat{x}_v \text{ whenever } v \in \mathbb{Z}^d \text{ and } \hat{x}_v \in \{0, 1\}\},$$

so that $\Phi(\hat{x})$ gives all the possible results from filling in $\square$ with 0 or 1. Let

$$\hat{X} = \{\hat{x} \in B^{\mathbb{Z}^d} : \Phi(\hat{x}) \subset X\}$$

denote the set of configurations such that $\square$ can be filled in arbitrarily with either 0 or 1 and still result in a valid configuration in X. It is known that if X is shift of finite type (SFT) then $\hat{X}$ is also SFT.

Question: If X is sofic is $\hat{X}$ sofic?

The answer is probably yes in general, and it is true when $d = 1$, by explicit construction, using follower sets in X. In general there is a natural candidate SFT cover for $\hat{X}$, but that does not work.

UPDATE: Shortly after the workshop, Ilkka Torma, student at University of Turku, solved this problem for $d = 2$ in the negative and developed the subject further.

Motivation: One wants to encode into a system which has some desirable constraints yet reserve some spots into which one can encode freely without violating the constraints. For example, the reserved spots could be used for error correction parity bits. Background and related work can be found in [12, 16].

2. (Karl Petersen)

This is a general question about how to recognize beta-shifts. As alluded to in the preceding question, sofic subshifts can in principle be recognized by seeing whether there are only finitely many follower sets. Also, the entropy should be that of an SFT, the logarithm of a Perron number. Analogously, hidden Markov chains (continuous factors of Markov shifts) can in principle be recognized by seeing whether the dimension of a certain algebraic object is finite—see the survey article [2]. None of these procedures may actually be easy to carry out. We ask just for some intrinsic characterization of the languages of beta-shifts.

3. (Klaus Schmidt)

Let X and Y be two-dimensional SFT's and $\phi : X \to Y$ a factor map. Must there exist a "nice" subshift $X_0 \subset X$ (e.g. sofic) such that $\phi|X_0$ is entropy preserving? (One cannot hope for finite to one). In dimension one, B. Marcus, K. Petersen, and S. Williams [14] showed that one can find a sofic X_0 with $\phi|X_0$ finite-to-one.

Mike Boyle: If one allows an ϵ increase in entropy, Angela Desai [3, 4] showed that one can find a sofic X_0.

4. (Mike Boyle)

Conjecture: If σ_A and σ_B are mixing $\mathbb{Z}$ SFT's with positive entropy, then there is an N such that for all $m, n \geq N$ there are SFT's $S \approx \sigma_A^m, T \approx \sigma_B^n$ (topological conjugacies), such that $ST = TS$ (S, T act on the same domain and commute).

Note that, for example, SFT's conjugate to σ_2 and σ_3 cannot commute, because of a periodic point obstruction.

5. (Uijin Jung)

Let $\phi : X \to Y$ be a factor map between shift spaces. If both X and Y are sofic shifts, then

$$S(\phi) = \{h(Z) : \phi = \phi_2 \circ \phi_1 \quad (\phi_1 : X \to Z, \phi_2 : Z \to Y)\} \subset [h(Y), h(X)].$$

It is known that $\overline{S(\phi)} = [h(Y), h(X)]$. This is also true for $\mathbb{Z}^d$ sofic shifts [11].

Suppose now that X is SFT, and let

$$S_0(\phi) = \{h(\phi_1 X) : \phi = \phi_2 \circ \phi_1, \;\; \phi_1(X) \;\; \text{SFT}\}.$$

Then $\overline{S_0(\phi)} = [h(Y), h(X)]$, and if X is a mixing SFT, $S_0\phi \subset [h(Y), h(X)] \cap \{\text{logarithms of Perron numbers}\}$.

Question 1: If X is a mixing $\mathbb{Z}$ SFT, is $S_0(\phi) = [h(Y), h(X)] \cap \{\text{logarithms of Perron numbers}\}$?

Question 2: If X and Y are $\mathbb{Z}^d$ SFT's, is $\overline{S_0(\phi)} = [h(Y), h(X)]$?

6. (Ronnie Pavlov)

There are theorems of the form "If A, then B is well computable. For example, "If X is SFT with some kind of mixing property, then its topological or measure-theoretic entropy or pressure is computable with some rate." Are there any theorems of the form "If A, then B is computable poorly" (in the realm of $\mathbb{Z}^d$ SFT's or $\mathbb{Z}^d$ sofic shifts)?

For example, can one prove any lower bound on the computability rate for any known "nice" $\mathbb{Z}^d$ SFT (not one explicitly constructed to have a bad entropy, as in [10], not the logarithm of a rational number)?

7. (Karl Petersen)

According to the Jewett-Krieger Theorem, every ergodic measure-preserving system is measure-theoretically isomorphic to a strictly ergodic homeomorphism of the Cantor set. There are many strictly ergodic systems that arise naturally from geometric, number-theoretic, or combinatorial constructions. How about exhibiting concretely Jewett-Krieger representatives for measure-theoretic systems that originate in other ways, for example, from cutting and stacking definitions? F. Hahn and Y. Katznelson [9] explicitly constructed positive entropy stricly ergodic systems. In the 1970's C. Grillenberger [6, 7] gave explicit constructions of strictly ergodic representatives for positive entropy, then K, then (with P. Shields, [8]) Bernoulli systems. We ask for more such explicit constructions. For example, the Pascal adic system (see [15]) has for its nonatomic ergodic measures the measures μ_α which correspond to the Bernoulli measures on the one-sided two-shift. Construct a Jewett-Krieger representative for the adic transformation with each of these measures. Being able to do so would help to understand the adic system as well as the process of making Jewett-Krieger representatives.

8. (Brian Marcus)

Given a $\mathbb{Z}^d$ subshift X, we want to "know" its entropy, $h(X)$. One interpretation would be to seek an algorithm which for each $\epsilon > 0$ produces approximations $h_-(X) \leq h(X) \leq h_+(X)$ with $h_+(X) - h_-(X) < \epsilon$ and h_+, h_- computable in time that is polynomial in $1/\epsilon$.

R. Pavlov [17] showed that this is possible if X is the $\mathbb{Z}^2$ hard-square shift. It is known for other examples [5] and [13].

Can you do better, or show that one cannot do better? One could try to replace computing time that is polynomial in $1/\epsilon$ by polynomial in $\log(1/\epsilon)$.

9. (Dominique Perrin)

Here are a few questions about noncommutative polynomials. The set of words on an alphabet A embeds into the free algebra $\mathbb{Z}\langle A\rangle$ generated by A (which also contains linear combinations of words on A). If X is a finite set of words, one can use it to make an SFT (for which X is the set of forbidden words), which might be empty. Call a set X of words *unavoidable* if any long-enough word has a factor (i.e., a subword) in X. What if X is not a set of words but a subset of $\mathbb{Z}\langle A\rangle$? Then X unavoidable might mean that the ideal generated by X has finite codimension?

A finite set $X \subset A^*$ is a *code* if every word has a unique decomposition into words in X. (Equivalently, the submonoid X^* generated by X is free, which implies that the algebra $\langle X \rangle$ generated by X is free.) For a subset X of $\mathbb{Z}\langle A \rangle$ this would correspond to the algebra generated by X being free. There is no known algorithm to test for this property, so a problem is to find out whether one exists. Is the property decidable? (Probably yes.) Is there a useful way to extend these ideas to $\mathbb{Z}^d$ shifts?

10. (Ronnie Pavlov)

For a $\mathbb{Z}$ subshift X on an alphabet A and a finite block w in the language $\mathcal{L}(X)$ of X, the *follower set* of w is

$$F(w) = \{x \in A^{\mathbb{N}} : wx \text{ is an admissible right ray in } X\}.$$

For each $n \in \mathbb{N}$ let

$$F_n = \text{card}\{F(w) : w \in \mathcal{L}(X) \cap A^n\}$$

denote the number of distinct follower sets of n-letter words in $\mathcal{L}(X)$. It is known that (F_n) is bounded if and only if X is sofic. It is a natural question whether (F_n) has to be nondecreasing; this has recently been answered in the negative with an example by Martin Delacourt.

Question: What can be said about the sequence (F_n)? For example, what is its growth rate for some subshifts? (Compare to the complexity function $p(n) = \text{card}(\mathcal{L}(X) \cap A^n)$, whose exponential growth rate is the topological entropy of X.) Is the sequence always not too far from monotonic, perhaps just oscillating between a few monotonic functions of n?

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