OPEN PROBLEMS SESSION 2
NOTES BY CHRISTOPHE REUTENAUER

1. Dominique Perrin

Following Alswehde: given a sequence of nonnegative integers $u = (u_1, u_2, \ldots)$, let its *Kraft sum* be $K(u) = \sum_{i \geq 1} u_i/2^i$. It is well-known that if $K(u) \leq 1$, then there exists a prefix set on the alphabet $\{a, b\}$ such that:

$$(*) \text{ for any } i \geq 1, u_i = |X \cap A^i|$$

(see the book *Codes and Automata* by Berstel, Perrin, Reutenauer, Cambridge 2010). For example, if $u = (1, 2, 0, 0, \ldots)$, then $X = \{a, ba, bb\}$. Recall that X is called *prefix* if no word in X is prefix of another word in X. Now, X is called *bifix* if X and its reversal (mirror) are both prefix sets.

Question: if $K(u) \leq 3/4$, does there exist a bifix code satisfying (*)?

The answer is known to be positive if $K(u) \leq 1/2$, and the bound $3/4$ is optimal (loc. cit.).

2. Reem Yassawi

Let $X = \{0, 1\}^{\mathbb{N}}$. Define the mappings T, M from X into itself by

$$T(1^n0x) = 0^n1x,\ T(1^\infty) = 0^\infty$$

$$M((01)^n1x) = (00)^n1x,\ M((10)^n0x) = (11)^n0x,\ M((01)^n00x) = (11)^n10x,$$

$$M((10)^n11x) = (00)^n01x,\ M((01)^\infty) = 0^\infty,\ M((10)^\infty) = 1^\infty$$

Then T, M are invertible transformations of X.

Question (Vershik and Solomyak): is the group generated by T, M free?

In other words, one must show that any nontrivial product $T^{j_1}M^{k_1} \cdots T^{j_p}M^{k_p}$ is not the identity. This can be shown to be true if $p \leq 4$ or if $\sum k_i \neq 0$. Note that $M(x) = T^{\phi(x)}(x)$ for some function (called a *cocycle map*) $\phi : X \backslash \{(01)^\infty, (10)^\infty\} \to \mathbb{Z}$.

Reference: Boris Solomyak, Anatoly Vershik, The adic realization of the Morse transformation and the extension of its action on the solenoid , Zapiski Nauchn. Semin. POMI 360 (2008), 70-91.

3. Natasha Jonoska

Let $f : X \to S$ be a cover of a sofic shift S by a shift of finite type X. Let $m(X, f)$ be the maximum of the cardinalities of the fibers $f^{-1}(s)$, $s \in S$. Let $min(S)$ be the minimum of $m(X, f)$, over all covers f.

Problem: compute $min(S)$.

It has been shown that if S is almost of finite type, or if $min(S) \leq 4$, then $min(S)$ is attained when X is the Fischer cover of S, and that this is not longer true for general sofic S. Evidently, S is a shift of finite type if and only if $min(S) = 1$.

Date: July 13, 2013.

Reference: Doris Fiebig, Ulf-Rainer Fiebig, Natasha Jonoska, Multiplicities of covers for sofic shifts, Theoretical Computer Science 262 (2001) 349-375.

4. Dominique Perrin

The Cerny problem: show that in a synchronized deterministic automaton, there is a synchronizing word of length at most $(n-1)^2$, where n is the number of states of the automaton.

Here, the automaton is a triple (Q, A, f), where Q is the set of states, A a finite alphabet and f a function from $Q \times A$ into Q. The function f is extended to a function $f : Q \times A^* \to Q$ by the formula $f(q, uv) = f(f(q, u), v))$. Each word $w \in A^*$ induces the function from Q into itself: $q \mapsto f(q, w)$. A word w is *synchronizing* if this mapping is of rank 1 (i.e., its eventual image is of size 1) and an automaton is called *synchronizing* if such a word exists.

It is easy to show that n^3 is a an upper bound on the length of the shortest synchronizing word, and nontrivial to show that $n^3/6$ is, too. The best known upper bound is cubic. For an automaton where some letter induces a full cycle (a *cyclic automaton*), the problem is solved. For an automaton where some letter induces an endofunction with a unique cycle (that is, its graph is weakly connected), $2(n-1)^2$ is a known bound. Note for each n, there exists a cyclic automaton such that the Cerny bound, $(n-1)^2$, is attained.

5. Valérie Berthé

Problem: let (f_1, f_2, f_3) be a triple of positive real numbers of sum 1. Find a sequence $u = (u_n) \in \{1, 2, 3\}^{\mathbb{N}}$ such that u has letter densities (f_1, f_2, f_3), that u has linear complexity (that is, for some constant C, the number of factors (i.e. subwords) of length n of u is $\leq Cn$ for any n), and that u has finite balance (that is, for some constant D, for each $i = 1, 2, 3$, for each prefix x of u, one has $||x|_i - nf_i| \leq D$, where n is the length of x and $|x|_i$ is the number of occurrences of the letter i in x).

The similar problem for 2 letters is solved by Sturmian sequences. It is known that the problem may be answered weakly, by replacing linear complexity by quadratic complexity.

6. Mike Boyle

Suppose that S is a one-sided $\mathbb{Z}$-mixing shift of finite type and T is a subshift.

Problem: when does there exist an embedding from T into S?

For two-sided shifts there is a simple answer: Krieger's embedding theorem. For the one-sided case, such an embedding ϕ must have a kind of automaton flavour: $x = x_0x_1x_2\cdots \mapsto y = y_0y_1y_2\cdots$. If $\phi(x) = y$, then the tree at x (obtained by the preimages of x in the shift) must embed into the tree at y. This must be compatible with an embedding of periodic orbits.

Special case of this problem: $S =$ full 2-shift, $h(T) < h(S) = \log 2$ and every point of T has at most 2 preimages. Does T embed into S?

One may assume wlog that T is an SFT. Indeed, if ϕ is an embedding of T into S with S SFT, then the local function that defines ϕ will also define an embedding into S of some SFT containing T.

7. Nishant Chandgotia

Let X be a d-dimensional nearest neighbour shift of finite type. Then

$$\Delta_X = \{(x, y) \in X \times X \mid x \text{ and } y \text{ differ at finitely any sites}\}$$

is known as the homoclinic relation. A Markov cocycle is a function $c : \Delta_X \longrightarrow \mathbb{R}$ which satisfies

(1) **Shift-invariance :** For all $x, y \in \Delta_X$ and σ a shift map (in any of the directions) $c(x, y) = c(\sigma x, \sigma y)$.
(2) **Cocycle condition :** For all $(x, y), (y, z) \in \Delta_X$, $c(x, y) + c(y, z) = c(x, z)$.
(3) **Markovian condition :** $c(x, y)$ is a function of $x|_{F\cup\partial F}$ and $y|_{F\cup\partial F}$ where x, y differ exactly at F.

The set of Markov cocycles denoted by $\mathcal{M}_X$ comes with a natural vector space structure and is in one to one correspondence with Markov specifications on X (a Markov specification is the collection of probability measures on configurations on finite sets of sites F, conditioned on a configuration on ∂F). If X has a safe symbol it follows that there is an algorithm to determine the dimension of $\mathcal{M}_X$ since every Markov random field whose support has a safe symbol is Gibbs with a nearest neighbour interaction (this is the Hammersley-Clifford Theorem). A shift space X is said to have the pivot property if for all $x, y \in \Delta_X$ there exists $x = x_1, x_2, \ldots x_n = y \in X$ such that x_i, x_{i+1} differ at a single site. For any nearest neighbour shift of finite type X with the pivot property the space $\mathcal{M}_X$ is finite dimensional.

Problem: is there an algorithm to calculate the dimension of $\mathcal{M}_X$ for a given nearest neighbour shift of finite type X with the pivot property?

8. Pascal Vanier

Let $G : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ be a 1-dimensional CA of neighborhood two and n the size of Σ. When G is non-surjective the configurations having no preimage always contain finite words (called orphans) that do not have a preimage by the local function that defines the CA. It is known that the size of the smallest orphan of G is always bounded by $(n + 1)^2$, and for any n it is possible to construct CAs whose smallest orphan is of size $2n-1$. Simulations on small numbers of states suggest that $2n-1$ is a (tight) upper bound.

Problem: Can you lower the upper bound on the size of the smallest orphan to $2n - 1$?

For a non-surjective CA, there exist two *different* words $w_1 = pc_1 \ldots c_k s$ and $w_2 = pc'_1 \ldots c'_k s$ which have the same image. These are called diamonds and k is their size. It has been proved that the size of the smallest diamond is bounded above by $2n\sqrt{n}$. it is not known whether this bound is tight, and it is in fact conjectured that $n - 1$ is an upper bound.

Problem: Find a tight upper bound on the size of the smallest diamond.

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