Shifts of finite type with nearly full entropy

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- This is a **nearest neighbor** (or **n.n.**) SFT: all forbidden configurations just pairs of adjacent letters

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- Note that $h(\mu) = h(\{1,\ldots,k\}^{\mathbb{Z}^d})$

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- Hard to give explicit description of MMEs in general

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- Examples of **Markov Random Fields** ("conditional independence of inside and outside"); more in Nishant's talk

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- Only such examples for $d = 1$; any irreducible $\mathbb{Z}$ SFT has unique MME

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- $\mathcal{I}_M$ is strongly irreducible, an extremely strong topological mixing property; used often to prove properties of $\mathbb{Z}^d$ SFTs.
- There are several conditions guaranteeing that a nearest neighbor $\mathbb{Z}^d$ SFT has a unique MME.

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  - In fact, we get more than just unique MME, also we know that $h(X)$ is computable to tolerance n^{-1} in time $e^{O(n^d)}$.

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  - In fact, we get more than just unique MME, also we know that $h(X)$ is computable to tolerance n^{-1} in time $e^{O(n^d)}$.
- Our proof gives $\alpha_d = d^{-17+o(1)}$.

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- $\log 2$ is optimal: $X = [1,n]^{\mathbb{Z}} \cup [n+1, 2n]^{\mathbb{Z}}$ has two MMEs.

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- The two components of A inducing distinct MMEs can interact, unlike $d = 1$ case

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- Ultimate goal: conjugacy-invariant checkable condition implying unique MME for all SFTs
  - This result is "closer" in the sense that it makes no explicit reference to safe symbols/allowed adjacencies, but it is still restricted to nearest neighbor SFTs...