Continuous-time Models in Corporate Finance

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OBJECTIVES

1) Option Pricing (1973—)
2) Corporate Finance (more) [impact of financial decisions on finns/value]

This course:

a) synthesis 6b 2
b) bridge gap btw 2 and 1

Questions to students:


LECTURE 1: INTRODUCTION

1) Elements of Corporate Finance

- Balance sheet of a firm = instantaneous picture
- what the firm owns (assets) and what the firm owes (liabilities)
- Throughout the textbook = simple case: 6 items

\[
\begin{array}{c|c}
\text{ASSETS} & \text{LIABILITIES} \\
\hline
\text{Proactive} & \text{Equity Et} \\
\text{Liquid reserves Mt} & \text{Debt Dt} \\
\hline
\text{Net worth} & \\
\end{array}
\]

- Income statement during period \( t \rightarrow t + dt \)

\[
\begin{array}{c|c|c}
\text{Net operating income} & \text{Profit/loss} & \text{Net change in liabilities} \\
\text{Financial income} & \text{Interest paid} & \text{Interest paid} \\
\text{Interest rate} & \text{Interest paid} & \text{Interest paid} \\
\text{For simplifying: debt structure} & \text{For simplifying:} & \text{For simplifying:}
\end{array}
\]

For the moment no taxes.
29) Pricing Risky Debt (Merton 1973)

This is an application of option pricing methods. Consider the simple case where $M(t) \equiv 0$, and debt consists of a single promised repayment $D$ at $t = T$ ($0 < T \leq 1$).

(i) $dX_t = (r - a - a_t)dt + \sigma dW_t$

NB: Asian options are protected by illiquidity.

Assumption: perfect secondary market, i.e., $S_T = \Phi(X_T)$

If $X_T > D$, they repay and get $X_T - D$

If $X_T < D$, they default and the holder gets $D$

Balanced value of debt at $t = 0$:

(3) $D_0 = E_0 \left[ e^{-rT} \min(D, X_T) \right]$

(Instead: $\min(D, X) = D - \max(0, D - X)$)

$\Rightarrow$ BS formula for a put option:

$D_0 = e^{-rT} D N(-x) + a_0 e^{-rT} N(x)$

$x = \frac{D}{\sigma \sqrt{T}}$; $x_0 = -x - \sigma \sqrt{T}$

NB: By definition $D_0 = e^{-rT} D$

Yield to maturity

(3) $\text{yield to maturity} = \frac{e^{-(r - \delta)T} D}{D_n}$

Comparative static analysis

(default spread to debt ratio)
23. The Modigliani–Miller paradox

Come back to the general case:

(1) \[ \Delta t + rM\Delta t = \Delta dt = dM + dL \]

not perfect zeroes

At \( t=0 \) firm raises \( D_0 \) in debt

and invests \( I + M_0 \)

Shareholder value \( SV_0 = E_0 = (I + M_0 - D_0) \)

Debt repay depends on \( \Delta t \) until \( T \)

and at \( T \) final repayment \( \min (A + M_T, D) \)

\[ D_0 = E_0 \left[ \int_0^T e^{-rt} \Delta t + e^{-rt} \min (A + M_T, D) \right] \]

\[ E_0 = E_0 \left[ \int_0^T e^{-rt} dL + e^{-rt} \max (A + M_T - D_0) \right] \]

\[ \Rightarrow \quad SV_0 = E_0 \left[ \int_0^T e^{-rt} (\Delta dt + dL) + e^{-rt} (A + M_T) \right] \]

\[ -I - M_0 \]

(4) \[ \Delta dt + dL = \Delta t + rM\Delta t - dM \]

\[ \int_0^T e^{-rT} (rM\Delta t - dM) = \int_0^T dL (e^{-rT} - M_0 - e^{-rT} A + rM_T) \]

\[ \Rightarrow \quad SV_0 = E_0 \left[ \int_0^T e^{-rt} \Delta t + e^{-rt} (A_T + M_T) + I_0 + M_0 \right] \]

\[ \Rightarrow \quad SV_0 = E_0 \left[ \int_0^T e^{-rt} \Delta t + e^{-rt} A_T \right] (I_0 + M_0) \]

(11) Irrelevance of financial policy \( \Delta dt \) (equity)

NB: MM also implies that risk management activities (reducing \( \rho \)) are at least unaltered (IE system)

or reduce shareholders value (IE system).
1) The Model

Shareholders are not cash constrained

\[ dAt = \beta Ate^{\mu} dt + 0 \cdot dW_t \]

Assume constant coupon \( c \) \( \Rightarrow \) ordinary model

\[ D_t = D(At) \quad E_t = E(At) \]

\[ \sum_{t=0}^{\infty} \frac{\Delta D(A) + \frac{c}{2} \Delta D(A) + c \Delta A}{\Delta A} = 1 \quad (1) \]

\[ D(A) = (1 - c) A \quad D(A) \sim \frac{c}{2} A, \quad A \to \infty \]

\[ D(A) = \mathbb{E}_A [ \int_0^\infty e^{-rt} \cdot c dt + e^{-rA} (1 - A)] \]

\[ D(A) = \frac{c}{2} - \mathbb{E}(e^{-rA}) \frac{c}{2} \mathbb{E}(e^{-rA}) - \mathbb{E}(e^{-rA}) (2) \]

\[ E(A) = \mathbb{E}_A [ \int_0^\infty e^{t \cdot \mu} (\Delta A - c) (1 - A) dt ] \]

\[ \sum \mathbb{E}(A) = \frac{c}{2} \mathbb{E}(A) - \frac{c^2}{2} \mathbb{E}(A) + c \mathbb{E}(A) \sim \mathbb{E}(A) (\beta - c) \quad A \to \infty \]

\[ \mathbb{E}(A) = \frac{c}{2} \mathbb{E}(A) + \frac{c^2}{2} \mathbb{E}(A) + \frac{c \mathbb{E}(A) - c}{A = A} \quad (3) \]
2. Asset Pricing

NB: General solution of linear equation

\[ E(A) = \alpha A + \beta A - \gamma \]  

where \( \alpha, \beta, \gamma \) are constants and \( A \) is an asset. 

Boundary condition at infinity \( A = 0 \)

NB: (2) \( E(e^{-\nu}) = (A)^{\infty} \)  

\( r, D, S \)  

3. Financial Decision 

Timing: 1) Review Assets C 
2) Market Rents D
3) Transferred Value AB

* Backward induction:

\[ AB \Rightarrow \max \left( \frac{S}{\gamma} \right) \]

(NE: \( r > 0 \))
Equity prices
for # values of $AB$

$$E(AB) = 1 + \gamma (AB - t) \frac{(t-s)}{2}$$

$$E(AB) > 0 \iff AB(1+\gamma) > \frac{t-s}{2} \iff AB > AB_{opt}$$

**Optimal $AB$ =** option like equity price ($\Delta = \gamma$)

### Pay with Trade-off Theory

- Default is always strategic.
  - Exercise if $L$ is option.
- Equity prices are convex functions of $A$.
  - Shareholders never want to manage risks (quite the opposite).

**STEP 1**

$$D_0 = \frac{\theta}{2} - \frac{(S - AB)(t-s)}{2A_{opt}}$$

$$SV_0 = E_0 - (I - D_0) = E_0 + D_0 - I$$

$$= (1-\theta) [\frac{S}{2} - AB + \frac{t-s}{2} - AB] \left(\frac{A}{A_{opt}}\right)^{\gamma_1}$$

$$+ \frac{\theta}{2} - \frac{A_{opt}}{2} (1-\theta) \left(\frac{A}{A_{opt}}\right)^{\gamma_1} - I$$

**STEP 2**

$$SV_0 = (1-\theta) A_0 - I + \theta \frac{s}{2} \left(\frac{1-A}{A_{opt}}\right)^{\gamma_1} \left[ (1-\theta) \left(\frac{S}{2} - AB \right) - \frac{t-s}{2} \right] + AB (1-\theta)$$

$$SV_0 = (1-\theta) A_0 - I + \theta \left[ \frac{1-A}{A_{opt}} \right]^{\gamma_1} - (\lambda-\theta) AB_{opt}^A + \theta \gamma_1$$

**Excess return analysis**
Then \( z = \left( \alpha - \frac{\mu}{\sigma} \right) \) is assigned to 4.

Maximise \( SV_0 \):

\[
SV_0 = (1-\theta)A_0 - I + \theta \left( \frac{1}{\theta^2} \right) \left[ AB - \frac{1}{\theta} \left( \frac{A_0}{\theta} \right) - \frac{\theta}{\theta^2} \right]
\]

\[
= (1-\theta)A_0 - I + \theta AB
\]

Max when

\[
\theta = \left( \frac{1}{\theta + \theta^2} \right) \left( \frac{A_0}{\theta} \right)
\]

\[
SV^*_\theta = (1-\theta)A_0 - I + \theta AB
\]

\[
= A_0 - I + \theta A_0 \left[ 1 - \left( \frac{\theta}{\theta^2 + 1} \right) \right]
\]

**NB:** Leverage decreases to \( A_0 \) increases with \( \theta \)
decreases with \( \alpha \)

**Correlation:** \( \gamma_1 \sim 1, \theta \sim -\frac{1}{3}, \alpha = -\frac{2}{5} \)

**Nominal Leverage:**

\[
\frac{\theta}{\theta} = \left( \frac{1}{\theta} \right) = \frac{2}{1 + \theta^2} = \frac{4}{5}
\]

**P.B. of M.O.** The M.O. predicts excessive leverage

**NB:** \( \theta = 0 \) (no taxes) \( AB_0 = 0 \)

**NB:** This is because shareholders are assumed to have no cash constraints