"MATHEMATICAL SOCIAL SCIENCES;" AN OXYMORON?

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Be honest. Deep down, most mathematicians probably accept "mathematical social sciences" as an oxymoron. I did — until after examining this area more seriously. Comfortable in my own research emphasizing the Newtonian N-body problem, I probably would respond to news that someone was working in, say, mathematical economics with a polite, "That's nice," where this noncommittal comment carried the tacit, private, yet universal translation among mathematicians, "What a 'wishy-washy' topic! Guess he can't do real mathematics." But approaches and events have significantly transformed fields. The social sciences have changed to the point that it now is appropriate to re-examine our beliefs. Of course, it always is possible to find disturbing examples. But when conducting a serious search of some of the social sciences, expect to discover that our thoughts are so dated that many (maybe not all) of them qualify as outright prejudices.

Times truly are changing. We just have to review what has been happening in our field of mathematics over the last thirty to forty years to recognize that this probably is the most exciting period in history to be a mathematician. New areas and topics are constantly developing. New approaches now permit mathematical advances which would have been met with skepticism only a decade or so earlier. We are experiencing a fascinating time when even major named problems keep falling.

Much more is happening. Mathematical excitement extends beyond the doors of a mathematics department to include a significant increase in the mathematical sophistication and advances of other disciplines. We all have a sense of this; all of us can point to another discipline — computer science, physics, biology, engineering — and recognize important, challenging mathematical concerns. But, the social sciences?

The point of these lectures is to indicate that the social sciences can be the source of a considerable mathematical interest. A major reason is that while the social sciences always have addressed deep and serious societal concerns, often the heaviest mathematical tool outside of statistics has been elementary calculus. This is changing; more and

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more of these issues are being expressed in mathematical terms where, for instance, it is not unusual to hear talks using Banach manifolds and while others worry about the correct topology on a function space. Nevertheless, there remain significant barriers. For instance, while it is possible to talk with an engineer, or a physicist, and have a reasonable notion of the central mathematical problems, it still requires added effort to interact with the social scientists. The problem is that they just don't talk like us — yet. So, another goal of these lectures is to indicate how to translate questions from the social sciences into mathematics; this is necessary because it is not obvious how to do this.

An explicit part of my agenda with these lectures is to encourage more mathematicians to seriously consider issues coming from the social sciences; let me assure you that you will find different and new mathematical issues. In the other direction, I also hope to encourage social scientists to appreciate the important gains that can result by using serious mathematics; I want to encourage the social scientists to seriously consider using this powerful approach. Because of these twin goals, my lectures, and these notes, are explicitly designed to address both audiences. For instance, the beginning of each section consists of examples which are intended to help develop intuition about the issues at hand. Then, toward the end of each section, there is a slightly stronger mathematical emphasis which is intended for the mathematicians. Nevertheless, I encourage the social scientists reading these notes to push on through this somewhat more technical material.

As for my comment about wanting to encourage mathematicians to look at the social sciences, the natural question, of course, is "Why bother?" An immediate answer invokes the common driving force which always has attracted the interest of mathematicians — the force of intellectual curiosity which has moved the field of mathematics to the enviable state which we enjoy today. By properly looking at the social sciences, one can find

- all sorts of new, unsolved mathematical problems
- which are mathematically challenging yet may be tractable, and
- where the solutions are relevant; they can be of distinct value.

While the mathematical social sciences is a surprisingly rich area, my discussion and choice of topics will, quite naturally, emphasize those issues and questions that have interested me.

Before turning to the discussion, let me offer a challenge. Can you find all functions f, g, and h which solve the functional equation

$$h[x + f(x - y)] = h(y) + g[h(x) - h(y)] \tag{0.1}$$

over the real line with the *minimal* possible smoothness? This functional relationship is nowhere near the standard kind normally encountered in mathematics, but it is a typical kind of relationship which arises from experimental work in mathematical psychology. An exposition supporting my comment while introducing other functional equations can be found in R. Duncan Luce's tribute to the mathematician János Aczél in a paper [18] entitled "Personal reflections on an unintentional behavioral scientist."

1. Mathematical Physical vs. Social Sciences

A way to identify one of the barriers which complicates communicating mathematical concerns with social scientists is to provide a quick, admittedly rough comparison of the mathematics of the physical vs. the mathematics of the social sciences.

For over a couple of millennia, mathematics and the physical sciences have enjoyed a sense of cooperation and interaction. This exchange has resulted in mutually rewarding symbiotic relationship where issues from the physical sciences create new challenges for mathematics, while new mathematical theories and approaches create new opportunities for the physical sciences. Risking an over simplification, the division of labor in the *Mathematical Physical Sciences* has the following feature:

1. *Creating models.* (With important exceptions, this primarily comes from the physical scientists. Mathematicians do create models, but many of them are intended to modify existing physical models into mathematically more tractable approximations.)
2. *Discovering basic mathematical properties of the models.* (With important exceptions, this primarily comes from mathematicians.)

Compare this status with the division of labor emerging from the *Mathematical Social Sciences.*

1. *Creating models.* (With important exceptions — such as where the model is a natural extension of the mathematics such as in game theory — this primarily and rightfully comes from the social scientists.)
2. *An identification of the basic mathematical issues, concerns.*
3. *Discovering basic mathematical properties of the models.* (Right now, most of this comes from the social scientists, but this will change. As more mathematicians discover the intellectual challenges of this area, expect that eventually most of these advances will come from mathematicians.)

Notice my caveat when describing the development of models in the social sciences. While it is admittedly tempting for mathematicians to try our hand at modelling, a word of advice is to *avoid this.* We are mathematicians; we are not social scientists. Much of our understanding of social science concerns is, at best, at the level of a "middle to high brow layperson;" it may be at a *New York Times* level, but this remains quite far removed from the level of a professional social scientist.

Much like the differences between amateur and professional mathematicians, when mathematicians start dabbling in the social sciences we do not have the background, intuition, insight, taste or sense of which variables, terms, and issues are important and which ones are not. Continuing with this comparison, while some amateur mathematicians may do quite well, most remain amateurs. Similarly, some social science models created by mathematicians might be successful, but many of the social science models that I have seen which were created by mathematicians have, thankfully, faded into oblivion. On the other hand, if the mathematics — if our strength — leads to a natural model, then go with it. But, to enjoy success in this area, a mathematician's emphasis should be placed on creating and understanding the underlying mathematics.

It should be directed toward creating a symbiotic relationship between mathematics and the social sciences.

In other words, while a sizeable portion of "mathematical social sciences" involves creating mathematical models, mathematician should move in a different direction which reflects our comparative advantage. Namely, after the social scientists identify important, pressing concerns, we should identify and extract the underlying and new mathematical structures. I will be more precise starting in the next section.

Notice the addition of a new point — the second one — in describing mathematical social sciences. To explain, the centuries of interactions between mathematicians and the physical scientists has generated a reasonably common language — with frustrating differences — which means that it does not take overly long to understand, in mathematical terms, major physical science problems. In comparison, it is fair to say that the mathematical sophistication of the social sciences dates to no more than the last couple of decades — at most the last half century. While some mathematical work dates to the nineteenth, or even the eighteenth century (e.g., Borda [6] and Condorcet [7]), most mathematical descriptions of the social sciences date from the 1970s on. Consequently, the current status of the social sciences is that the translation of basic issues and concerns into mathematical terms remains the responsibility of the mathematician.

This "translation factor" makes the mathematical social sciences more difficult, but it also provides a delightful challenge. For me, this "translation" question proves to be a creative source of the mathematical challenge of the social sciences. But, from my experience, a key in understanding how to address this issue is:

> Don't listen to what the social scientists are saying, listen to what they would like to say if they understood the real power of mathematics.

This factor will be emphasized in some examples given next.

2. Symmetry galore![1]

It is time to move from this somewhat abstract, philosophical discussion to a more practical tone motivated by concrete examples. To do so, I use one of my favorite examples (e.g., see the first chapter of Saari [44]) which involves a committee of 15 preparing for a party. To save money, they decide to select only one beverage. The individual preferences are

- 6 prefer Milk to Wine to Beer (denoted as $M \succ W \succ B$),
- 5 prefer $B \succ W \succ M$, and
- 4 prefer $W \succ B \succ M$.

A simple vote shows that this committee prefers

$$M \succ B \succ W \text{ with the tally } 6 : 5 : 4. \tag{2.1}$$

After the committee arrives at the beverage store to order a keg of Milk, they discover that Beer was not an option because the economists arrived earlier and bought all of the beer. But this information appears to be irrelevant because Beer was in second place while the two available choices, Milk and Wine, are in first and last. Choosing between Milk and Wine would not change anything. Or would it?

[1]Many of the results of this section come from my books Saari [47, 48].

Check. If these alternatives are compared pairwise, we end up with the outcome

Outcome	Tally
$W \succ M$	$9:6$
$B \succ M$	$9:6$
$W \succ B$	$10:5$

(2.2)

In other words, these voters prefer either other option to "first-place" Milk, and they prefer "last place" Wine to Milk or Beer. The pairwise vote suggests, then, that these voters want the precise opposite of the Eq. 2.1 outcome given by the standard plurality vote; it appears that they really prefer the reversed outcome of

$$W \succ B \succ M. \tag{2.3}$$

It is interesting to observe that this reversed conclusion cannot be dismissed as a peculiarity because, in each case, the winner wins with decisive support of at least 60% of the vote.

Do you want Beer? No problem; just use a runoff. Here, Wine comes in last place in the $M \succ B \succ W$ first election, and Beer beats Milk in the runoff. In other words, with no change in the voter preferences,

- *Milk* wins with the standard plurality election,
- *Beer* wins with the standard runoff procedure, and
- *Wine* wins when alternatives are compared pairwise.

Stated in starker terms,

> *rather than reflecting the views of the voters, an election outcome may more accurately reflect which decision procedure was used.*

Rather than a hypothetical example, the election phenomenon illustrated by this beverage example can, and has, occurred in many actual elections. For examples, I refer to my books Saari [44, 47, 48] as well as those by Nurmi [30, 31] among many, many others.

2.1. **Taking a mathematical perspective.** This example illustrates the kind of problem faced by social scientists; it is bothersome; it is mysterious; and clearly any explanation or resolution of the difficulty would be useful. So, what should be done? What does this example mean?

Answers to these questions and typical reactions to the example coming from social scientists tend to reflect the tools they know about and the reality they face. Yes, they are very troubled by the implications of such an example, but many of them tend to change the subject by describing legislative processes developed to circumvent (or, maybe only hide) such difficulties, or to express their hope that such problems are merely isolated anomalies.

A mathematical perspective, however, is to accept that this example captures an important social science issue and then pose the problem in a mathematical framework which suggests several standard mathematical questions. To illustrate, the example describes a mapping from a space which lists all possible preferences of individuals — these are called *profiles* — to a product space (one for the outcome for each of the three election procedures) of election rankings or outcomes. So, if $\mathcal{P}^N$ lists the transitive ways to rank the N alternatives $\mathcal{A}^N$, then this particular 15-voter example defines

$$F : (\mathcal{P}^3)^{15} \to \mathcal{P}^3 \times [\mathcal{P}^2]^3 \times \mathcal{A}^3. \tag{2.4}$$

(The first $\mathcal{P}^3$ component of the image space has the plurality ranking, the second $[\mathcal{P}^2]^3$ component has the three pairwise rankings, and the third $\mathcal{A}^3$ specifies the winner of the runoff.)

Expressing the example in terms of Eq. 2.4 leads to several natural questions for mathematicians; questions which constitute new issues for the social scientists. To illustrate, let me list some of these mathematical questions and then immediately follow (in brackets) by describing the associated social science concern.

1. What is the image of F? [Is it possible to characterize all possible voting paradoxes that could ever occur?]
2. For any $\mathbf{a} \in \mathcal{P}^3 \times [\mathcal{P}^2]^3 \times \mathcal{A}^3$, what is the structure (i.e., convexity structure, measure, etc.) of the set $F^{-1}(\mathbf{a})$? [How likely are paradoxes; how dominant are outcomes with no conflict? Can other problems arise with election rankings?]
3. Do theses answers change with the value of N? Suppose the exponent "15" is replaced with an arbitrary integer; what can happen now? What about different linear mappings and their combinations? [Can we characterize all possible paradoxes that can occur with any number of voters, any number of candidates, and for any choice of voting procedures?]
4. Voting is an "aggregation" procedure; what happens with related, but nonlinear mappings? [Do these voting paradoxes extend to affect other social science phenomena, or are they specific to voting?]

While these mathematical questions may be difficult to answer, they introduce new mathematical structures into the analysis. Moreover these standard mathematical issues (e.g., characterize the domain, the range) already generate a more ambitious research program than most social scientists would consider possible. Without question, answers to any of these questions would be welcomed. In other words, let the social scientist dictate the "taste" by describing what is important and what is not. By reformulating these issues in a mathematical manner, it is possible for mathematicians to raise crucial, important and new questions.

2.2. **Some answers.** First, let me address the question raised above whether this peculiar behavior — of the type illustrated by the beverage example — is particular to voting, or whether it extends to other aggregation methods. Intuitively, the answer is clear. The voting procedures described in the beverage example are linear, so not only should we should expect the inconsistencies in outcomes to extend to non-linear methods, but we should expect the nonlinearities to introduce even more problems.

To be more precise in terms of an example, consider the following *decision analysis* problem posed in terms of a game of chance. The game is to select one of two urns and reach inside to randomly select a ball. If a red one is selected, you win a substantial prize; if a blue one (the other color) is selected, you pay a substantial penalty.

Now suppose you have an advantage; you discovered that urn one contains a higher fraction of red balls than urn two; i.e., the probability of selecting red from urn 1 is larger than the probability of selecting red from urn 2 expressed as

$$P(R|1) > P(R|2).$$

The decision choice is obvious; select urn #1.

To convert this example into an aggregation procedure, suppose there is a second set of urns $1'$ and $2'$. Again, suppose that

$$P(R|1') > P(R|2').$$

Again, the optimal decision is to select urn $1'$.

The aggregation problem comes from combining the contents of 1 and $1'$ to create urn $1''$, and those of 2 and $2'$ to create urn $2''$. Which urn should be selected now; urn $1''$ or urn $2''$?

Just by the fact the issue is being raised, it is reasonable to suspect that the natural answer of $1''$ need not always be correct. That this can happen is indicated in Fig. 1 where the notation indicates that urn 1 has 900 red balls and 1500 blue ones, etc.

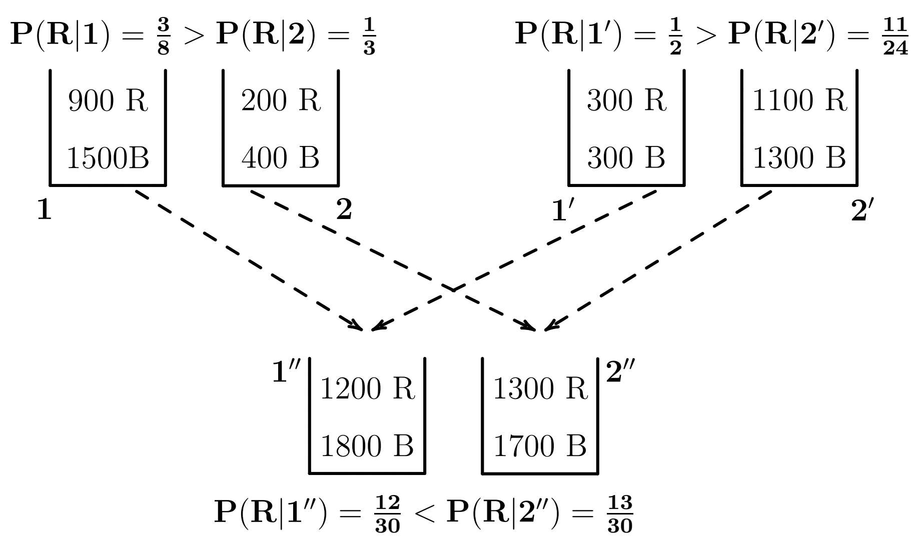


Fig. 1. Aggregation paradox

Again, this is no hypothetical example; it can and does cause problems. In statistics, this phenomenon known as *Simpson's Paradox* (Simpson [61]) seemed to have been first observed in data for health problems. An interesting, more recent example is when this mathematical peculiarity plagued the state of California in their **API** program which is intended to improve the school system. Here, the two sets of initial urns (1, 2) and (1', 2') identify where $P(R)$ indicates the fraction passing particular education standards. Thus, the $P(R|1) > P(R|2)$ and $P(R|1') > P(R|2')$ relationships state that both groups improved from last year to this year. According to the state incentive program, schools with this data should anticipate substantial rewards for the teachers. But the aggregate conclusion $P(R|1'') < P(R|2'')$ asserts that the school did poorer! As such, the state program placed this school on a "to-be-watched-with-concern" list. This was not an isolated phenomenon; during the first year of the California program, this Simpson paradox affected 70 schools! You can imagine the fuss created by this mathematical quirk! (See Saari [48]).

For us, the mathematical questions associated with Simpson's paradox are obtained by posing the problem in an Eq. 2.4 formulation. Again, the associated mathematical questions are similar; e.g., other than Simpson's paradox, what else can happen? What is the likelihood of these different behaviors, and how can they be avoided? But rather than exploring these questions here, the main point is to point out that voting problems are closely connected to issues arising with any class of aggregation methods whether they come from probability, statistics, voting, economics, etc. What we must expect, and it is proving true, is that the results discovered from voting extend to these other areas — except that more can happen with the nonlinearities. (For a taste of what can happen, see Saari [38].)

2.3. **How bad can it get?** To indicate the research rewards which can result from using this approach, let me indicate the kinds of results which can be obtained. More detailed descriptions of everything described in the remaining subsections of this Section 2 can be found in Saari [47] and the indicated references.

First, it is clear that a reason the plurality election method allows election oddities is that the procedure only registers a voter's top choice. In the beverage example, for instance, the plurality vote of "voting for one" loses all of the valuable information that Milk is the bottom choice for 60% of the voters. This weakness was noticed by the French mathematician J. C. Borda [6] in 1770,. His observation and comments attracted the attention of other mathematicians, such as Condorcet [7] and Laplace, and resulted in some excellent research providing insight into these problems.

A way to register more information about the voters' preferences is to use what Riker [34] calls *positional voting.* This is where ballot is tallied by assigning specified points to how each candidate is positioned (i.e., ranked) on the ballot. These points are given by a *voting vector*

$$\mathbf{w}^n = (w_1, w_2, \dots, w_n = 0), \quad w_i \geq w_{i+1}.$$

For instance, the plurality vote is where each voter votes for one candidate, so it is given by $(1, 0, \dots, 0)$. A method often used in academic departments, where we are asked to vote for k candidates, is given by $w_1 = w_2 = \dots = w_k = 1; w_{k+1} = \dots = w_n = 0$. The method proposed by Borda, now known as the Borda Count, has

$$\mathbf{w}^n = (n-1, n-2, \dots, 1, 0).$$

2.3.1. *Examples.* When confronted with a continuum of possible methods, a mathematician wants to know whether the choice makes any difference. In other words, what happens should different methods be used. The following example indicates that we should expect worrisome conclusions. For the ten-voter profile

	Preference		Preference
2	$\mathbf{A} \succ B \succ C \succ D$	2	$\mathbf{C} \succ B \succ D \succ A$
1	$\mathbf{A} \succ C \succ D \succ B$	3	$\mathbf{D} \succ B \succ C \succ A$
2	$\mathbf{A} \succ D \succ C \succ B$		

the election outcomes are as follows

Voting procedure	**Winner**	**Voting procedure**	**Winner**
Vote for one:	**A**	Vote for two:	**B**
Vote for three:	**C**	Borda:	**D**

Thus, this innocuous appearing ten-voter profile, allows *any of the four candidates to win just by using an appropriate voting procedure.* Moreover, instead of the problem being caused by using esoteric voting procedures, they are commonly used.

Other examples can be constructed to exhibit this phenomenon for any value of n. To illustrate, the five-candidate profile

No.	**Preference**	**No.**	**Preference**
3	$A \succ B \succ C \succ D \succ E$	2	$D \succ C \succ E \succ A \succ B$
1	$A \succ C \succ E \succ D \succ B$	1	$E \succ A \succ C \succ D \succ B$
2	$A \succ E \succ C \succ D \succ B$	3	$E \succ B \succ D \succ A \succ C$
2	$C \succ B \succ D \succ E \succ A$		

exhibits the same election behavior because by voting for one candidate A wins, by voting for two candidates B wins, by voting for three candidates C wins, by voting for four candidates D wins and by using the Borda Count E wins. Again, election outcomes may more accurately reflect the choice of a procedure rather than what the voters actually want.

While examples are illustrative of what might occur, a mathematician prefers a general result. The following is one possibility.

Theorem 1. *(Saari [37, 47]) For $n \geq 3$ candidates and for any k satisfying $1 \leq k \leq n! - (n-1)!$, there is a profile where precisely k different and strict (i.e., no ties) election outcomes arise when the ballots are tallied with different positional methods. It is impossible to create an example with $k > n! - (n-1)!$.*

As a special case, this result means that with only ten candidates — about the number starting in some presidential primaries in the US or in French presidential elections — it is possible to have millions of different election outcomes just by changing the positional method. Even worse, once $n \geq 4$, examples can be constructed where each candidate is in first place with some methods, second with others, ... , and last place with still others. Also, the conclusion is robust in the sense that after imposing a natural topology on the space of profiles, open sets of examples can be found. So, which candidate is the *real* choice of the voters?

2.3.2. *Outline of mathematics.* To suggest the mathematics behind this theorem, first express all voting vectors so that the lead value is $w_1 = 1$. As an example, the four-candidate Borda Count vector $(3, 2, 1, 0)$ now becomes $(1, \frac{2}{3}, \frac{1}{3}, 0)$. With this normalization, any positional method can be expressed as a convex combination of the "vote for k" methods. For instance, with $n = 4$, the Borda Count is the baricentric point

$$(1, \frac{2}{3}, \frac{1}{3}, 0) = \frac{1}{3}[(1,0,0,0) + (1,1,0,0) + (1,1,1,0)].$$

In an obvious manner, an n-candidate election outcome can be expressed as a point in R^n_+; e.g., if the x, y and z axes are assigned, respectively, to Anni, Bev, and Candy, then $(50, 24, 30)$ is the election tally for a particular procedure. Exploiting the linearity of positional methods both in profiles and in the weights, it turns out that the tallies of a profile for all election outcomes are in a particular convex hull which is defined by $n-1$ vertices. The kth vertex is given by the profile's election tally for the "vote for k" method, $k = 1, \dots, n-1$. Conversely, each point in this *procedure hull* is the outcome for one voting vector. Thus, for a specified profile, the geometry of the procedure hull identifies all possible election outcomes. The conclusion of the above theorem is the inverse problem; it is to determine which of the possible geometric positionings of these convex hulls within R^n_+ can be supported by a profile.

It is of particular interest that this theoretical tool of a procedure hull has proved to be a pragmatic tool of analysis. Indeed, A. Tabarrok [69] and Tabarrok and L. Spector [70] used this methodology to obtain a new insight into the 1992 US presidential election among Bush, Clinton, and Perot, and the 1860 US presidential election which precipitated the US Civil War. By use of the procedure hull, they now can identify all possible outcomes that could have occurred in these elections had different positional methods been used.

2.3.3. *Likelihood.* Another issue remains; can these conclusions be dismissed as reflecting highly unlikely settings? The answer is no; the troubling conclusion is that inconsistencies of election outcomes dominate. For instance, Maria Tataru and I developed a method [54] showing that under conservative assumptions on the probability distribution of profiles, with only three candidates there is about a 69% chance of the election ranking changing with the choice of an election procedure. The probabilities get worse with more candidates. Subsequently, different combinations of Merlin, Tataru, and Valgones used this geometric approach to obtain other results describing the likelihood of different events. (See, for example, [27, 73].)

2.3.4. *Extensions.* Above I asserted that all of these voting results would extend and provide new conclusions for other aggregation methods. This is the case. For instance, by using Thm. 1, a similar result was developed by D. Haunsperger [13] for non-parametric statistical approaches. In a different direction, this result also was used, independently, by Laurelle and Merlin [17] and by Saari and Sieberg [51] to develop a method of measuring political power in terms of game theoretic methods. Again, the conclusions are discouraging. Again, it was the mathematical perspective which posed the questions which allowed these conclusions.

2.4. **Dropping candidates.** To motivate the next issue, suppose in search for someone to fill the one tenure track position from among the candidates $\{A, B, C, D\}$, the preferences of the selection committee are

	Preference		Preference
3	$A \succ C \succ D \succ B$	2	$C \succ B \succ D \succ A$
6	$A \succ D \succ C \succ B$	5	$C \succ D \succ B \succ A$
3	$B \succ C \succ D \succ A$	2	$D \succ B \succ C \succ A$
5	$B \succ D \succ C \succ A$	4	$D \succ C \succ B \succ A$

The plurality election outcome of $A \succ B \succ C \succ D$ with a 9:8:7:6 tally is immediate. Presumably an offer would be made to A even if D calls to let us know that she accepted a position in a much better school. But, is it? Check and you will see that if these same voters vote over the pool of $\{A, B, C\}$ the outcome is the reversed $C \succ B \succ A$. In fact, if any candidate, or any pair of candidates, are dropped, the new outcome has an outcome coming from the reversed $D \succ C \succ B \succ A$ ranking. So, do the members of this department really prefer A, or do they prefer D?

How bad can it get? Is this behavior associated only with plurality voting, or does it plague all positional methods? The answer provides more information about whether we should trust election outcomes. In this theorem, a statement is made about an implied space of positional voting methods. Think of this as being a subset of the appropriate Euclidean space where the entries are obtained by listing the positional methods for each subset of candidates one after the other to form a huge vector.

Theorem 2. *(Saari [36, 47].) For $n \geq 3$ candidates, rank them in any way. Then, for each of the $\binom{n}{k}$ sets of k candidates, $2 \leq k < n-1$, rank the candidates in any desired manner. For each set of candidates, specify a positional voting method to be used to tally the ballots. With the exception of a lower-dimensional algebraic variety of positional methods, a profile can be constructed so that for each subset of candidates the specified election outcome arises when the ballots are tallied with the specified positional method.*

What a chaotic conclusion! It states that with the ranking of $A \succ B \succ C \succ D \succ E$ of five candidates, you can use a random number generator to specify rankings for the five subsets of four candidates, the ten subsets of three candidates, the ten subsets of pairs, and there exists a profile where these are the sincere election outcomes.

This conclusion holds not just for the maligned plurality method, but for almost all ways to tally ballots! It is interesting that this "seemingly random behavior" conclusion shares similarities with chaotic dynamics, *and* the proof was developed by modifying mathematical notions coming from chaos. (An exposition in is Saari [38, 47].) Readers familiar with chaos know that a key involves bifurcation points; in voting these points are given by ties. Replacing some of the analytic tools from chaos are algebraic tools involving symmetry orbits.

Immediate questions involve the algebraic variety of methods. Does choosing a method from this variety improve the consistency of outcomes? What is the structure of this algebraic variety? The second question goes beyond the intended nature of these lectures, but some of the mathematical ideas are indicated below and in [47]. For the first question, the Borda Count, thanks to its symmetry (of having a fixed difference between subsequent weights) is in this variety for all choices of $n \geq 3$.

To show the difference the Borda Count makes, let $\mathcal{D}^6_P$ be a listing of all of the different plurality election rankings which can occur over all subsets of six candidates. Namely, an entry of $\mathcal{D}^6_P$ specifies a listing of election rankings which can occur with the plurality vote for all six candidates, the six rankings which can occur with the six subsets of five candidates, etc. Similarly, let $\mathcal{D}^6_B$ be the same for the Borda Count. If $|\mathcal{D}^6_P|$ is the number of these plurality outcomes, a large value indicates many inconsistencies and paradoxes. So, the goal is to compare $|\mathcal{D}^6_P|$ and $|\mathcal{D}^6_B|$; a small difference means that it really does not matter which voting method we should use, while a large difference means that the Borda Count would provide more consistencies. The answer is that the Borda Count outcomes are significantly more consistent as

$$10^{50}|\mathcal{D}^6_B| \leq |\mathcal{D}^6_P|.$$

Thus, a listing showing an election inconsistency with the Borda Count can be modified in 10^{50} different ways to indicate inconsistency problems which arise with the plurality vote. The size of the numbers show that even by using all of the faster computers in existence, it would be impossible to list these paradoxes.

2.5. **Finding a universal description.** Once a mathematician learns about the above results which characterize all possible voting paradoxes which can occur, the next natural question is to identify the underlying mathematical structures which cause these conclusions. Re-expressing this mathematical goal in terms of the social sciences, the new objective is to explain the reason for each and every voting paradox, to be able to construct examples illustrating each and every paradox. This objective is outlined in Saari [47, 50] in an intuitive manner, so let me offer just a couple of added words.

The underlying structure comes from symmetry groups. To see why, notice that starting with the $A \succ B \succ C$ preferences, the orbit over all permutations of these three objects defines all possible transitive rankings. This suggests that algebraic orbits of subgroups must be important. The notion of which subgroups of the permutation group are interesting comes by recognizing that subsets of candidates are involved.

Before starting, notice that the permutation $\sigma_{A,B}$ of interchanging A and B has a related effect on how several of the pairs of candidates should be ranked, but the appropriate symmetry depends on what ranking is $\sigma_{A,B}$ applied to. For instance, $\sigma_{A,B}$ applied to $A \succ B \succ C$ induces the identity map on $B \succ C$ and $A \succ C$ and the flip on $A \succ B$. But applying $\sigma_{A,B}$ to $A \succ C \succ B$ generates a flip on all three pairs. It is this difference, which is familiar to algebraic topologists and a version of what is called the wreath product, which the source of many of the voting paradoxes.

To explain, there are certain symmetry subgroups which give one kind of behavior for one set of candidates, and another for different subsets. A simple example is just the Z_2 action, or flip. This creates an orbit of the type

$$A \succ B \succ C,\ C \succ B \succ A \tag{2.5}$$

where the orbit for each pair also consists of a ranking and its flipped ranking. This ranking sure indicates a tie outcome.

Such a complete tied outcome does occur for the pairwise elections, but not for all positional methods. Indeed, with $(1, s, 0)$, it is clear that A and C each receives one vote, but B receives $2s$ votes. So, the tie arises only if $2s = 1$, or if the Borda Count is used. Any other positional method must admit profiles where the positional and pairwise outcome conflict.

The next natural group action is Z_3 giving rise to

$$A \succ B \succ C,\ B \succ C \succ A,\ C \succ A \succ B.$$

Here, it is arguable that the outcome should be a tie; indeed, this is the outcome for all possible positional methods. But this group action is precisely the kind described above where what happens to a pair depends on what ranking the Z_3 is acting on. Thus, we have two rankings with $A \succ B$, and one with $B \succ A$; two with $B \succ C$, and one with $C \succ B$; two with $C \succ A$, and one with $A \succ C$. This means that rather than a tie, that the pairwise election outcomes define a cycle where

$$A \succ B,\ B \succ C,\ C \succ A,\ \text{all by tallies of } 2:1.$$

The mathematical method of attack, then, is to consider the orbits of various symmetry subgroups. For some sets of candidates, the orbit structure defines a natural configuration of voter preferences which should end up in a complete tie. Now look at the consequences of this orbit structure on the subsets of candidates. If it does not define a tie, we must expect that "enough" components of this type will cause an inconsistency in election rankings. The surprise is that this is the explanation for all possible positional voting paradoxes.

2.6. **Creating examples.** If the last comment that symmetry groups are the cause of all of the positional voting paradoxes, then it should be easy to create examples illustrating any possible paradox. To show that this is the case, start with the profile

Number	**Preference**	**Number**	**Preference**
5	$A \succ B \succ C$	2	$B \succ A \succ C$

(2.6)

There should be no argument; since everyone had C bottom ranked, the outcome is between A and B. Thus, the natural outcome should be $A \succ B \succ C$.

To create inconsistencies, we need to add certain amounts of the orbits described above. Now, the Z_3 orbit has no effect on positional methods, so add x units of it to change the pairwise election so that B, not A, wins all pairwise elections. That is, add x voters preferring each of

$$B \succ A \succ C, \quad A \succ C \succ B, \quad C \succ B \succ A. \tag{2.7}$$

With the Eq. 2.6 terms, the pairwise tallies become

Pairs	**Tallies**
$A:B$	$5+x:2+2x$
$B:C$	$7+x:2x$

Thus, any x value satisfying $2x+2>5+x$ and $7+x>2x$ suffices; $x=4$ is one value. Notice, adding this configuration to the Eq. 2.6 profile has no effect on the positional rankings.

Now use the Z_2 or reversal rankings. These configurations have no influence on the pairwise rankings, but they affect all positional outcomes except the Borda Count. So, add enough of these terms to change the plurality outcome to $C \succ B \succ A$. Namely, add the configuration

Number	**Rankings**
y	$C \succ B \succ A$ and $A \succ B \succ C$
z	$C \succ A \succ B$ and $B \succ A \succ C$

(2.8)

Along with Eq. 2.6, the new plurality outcome is

$$A:B:C \text{ with the tallies } 5+y:2+z:y+z.$$

Thus, choosing y and z so that

$$y + z > 2 + z > 5 + y,$$

where $y = 3$, $z = 7$ suffice, an example is constructed. Thus, by combing the three component parts of Eqs. 2.6, 2.7, 2.8 along with $x = 4, y = 3$, and $z = 7$, a desired example emerges. In other words, the symmetry decomposition makes the construction of illustrating examples mathematically trivial.

3. Singularity theory and departmental meetings[2]

In the last section, I showed how election outcomes may more accurately reflect which election procedure which was used rather than the views of the voters. To make this more dramatic, consider the following 30 voter, eight-candidate profile

No.	**Ranking**
10	$A \succ B \succ C \succ D \succ E \succ F \succ G \succ$ **H**
10	$B \succ C \succ D \succ E \succ F \succ G \succ$ **H** $\succ A$
10	$C \succ D \succ E \succ F \succ G \succ$ **H** $\succ A \succ B$

(3.1)

The data strongly supports the notion that C is the group's favorite. Without question, the group's least favorite candidate is H because *everyone* prefers C, D, E, F, G to H. The disturbing fact is that in spite of this preferences, H could be elected by this group, and she could be elected in a manner where everyone would accept that she is the group's overwhelming favorite. Before I show how, let me offer the challenge to try to find out how this is done.

The approach is just the agenda of Fig. 2. By computing the results with the above profile, it turns out that each election is determined by unanimity or by the overwhelming setting where the winner receives two-thirds of the vote. With such outstanding votes these voters most surely would treat the conclusion as reflecting the wishes of the groups — even though each and everyone of them find the conclusion to be the inferior choice. Namely, although H is *not* who they want, H is who they get.

What makes this agenda bothersome is that it is of the kind which can arise in departmental discussions. Notice, some of the stronger choices are discussed in the beginning, dismissed, and then compared with later ones. The weakest choice is compared last. This kind of election behavior should force us to wonder whether, in sincere departmental discussions, we choose badly only because of the order in which topics are introduced. This is bothersome.

Where does this example come from? If you check Eq. 3.1, you will notice that these are just the first three terms of a Z_8 orbit. Indeed, it takes only three terms from a Z_k, $k \geq 3$, orbit to create a cyclic voting outcome. When this occurs, an appropriate scheduling of the choices allows any candidate to win. Incidentally, these Z_k orbits of data are responsible for the pairwise ranking problems which arise in probability, statistics, economics, Arrow's Theorem [2] from decision analysis, Sen's Theorem [58] which

[2]The thrust of this section is to describe seminal results found by McKelvey [23, 24] as well as extensions of his results found by my former Ph.D. student Maria Tataru [71, 72]. It concludes with results from my research on the stability of "cores" as described in Saari [43].

has bothered some economics, political scientists, philosophers, etc. (A description is in Saari [48].)

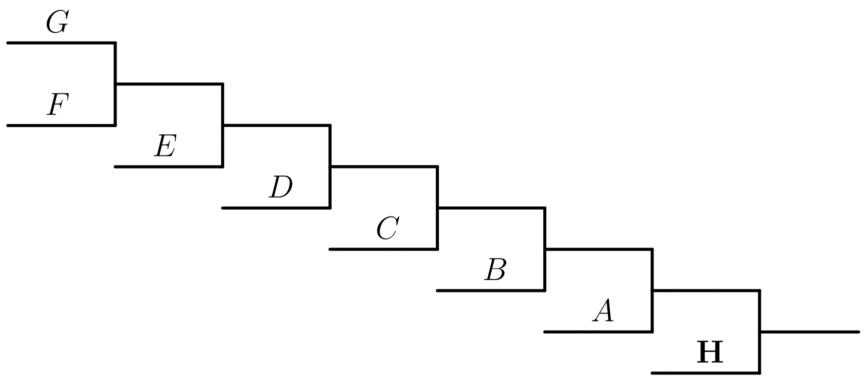


Fig. 2. An agenda

3.1. **Other symmetries.** I am going to build on this symmetry by showing how versions of it arise when considering discussions of the type found in departmental meetings. As part of the discussion, it will turn out what many of us always have suspected; departmental politics is a natural example of *true dynamical chaos.*

The difference from the above is that rather than specified alternatives that will be voted upon, the voters can propose, or shade by slightly altering proposals to something which is more preferred. A way to think about this is to recall actual examples; e.g., a draft proposal put forth in the department about, say, the requirements for earning a Ph.D., or even the requirements for making long distance professional phone calls.

These proposals combine several issues. But, this is not anything of particular surprise because it is difficult to think of any vote which does not involve multiple issues. A person's vote for a president, or political party, may involve an aggregation of views about women's rights, foreign policy, and so forth. A vote for a departmental chair may involve the abilities of each candidate to carry out the responsibilities *and* how the selection of that person will affect the future of my particular research group.

3.2. **Issue space.** The issues, whatever they are, define the coordinate axes of Fig. 3a. The more issues, the larger the dimension of what is called "issue space." Each voter's personal and idealized stand on these issues is represented by their "ideal point" depicted by the bullets in Fig. 3a. For instance, the difference in the stand of voter's 2 and 3 is characterized by which coordinate is larger, so this represents where they place different value, or emphasis on the issues. A proposal also defines a point in this space; the closer a proposal point is to an voter's ideal point, the more the voter likes it.

Figure 3a captures a fairly standard division of voter views; they are clustered with a slight difference of opinions. Of course, compromises must be made to reach a final outcome, but it is reasonable to accept that the final outcome will be somewhat centered near the baricenter of the defined triangle. Indeed, let me ask the reader to speculate about the positioning of an appropriate proposal.

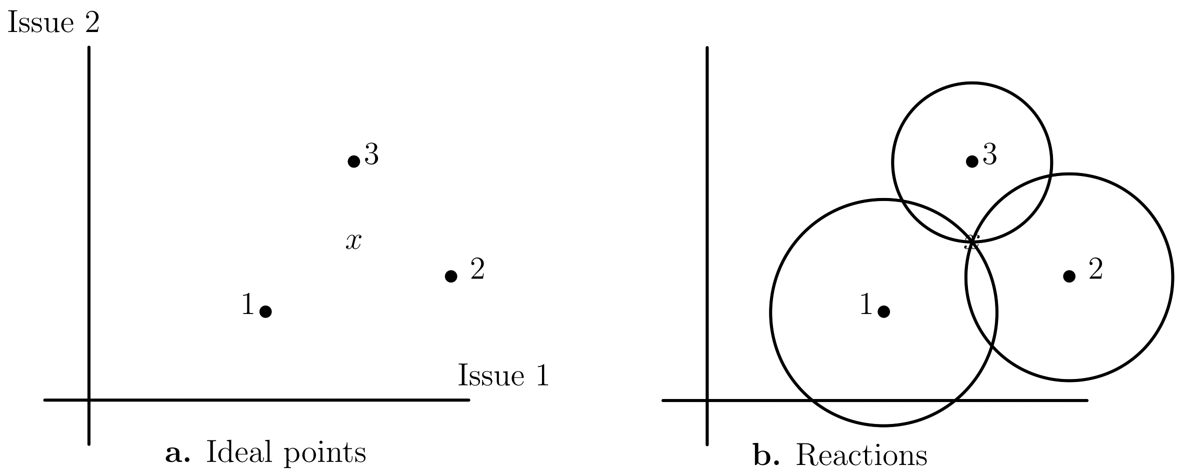


Fig. 3. Multiple issues

For a first answer, the location of x in Fig. 3a seems to be a reasonable compromise. One might suggest modifications; since the position of x is closer to 3's ideal point, it might be more equitable proposal to locate the proposal somewhat lower in the picture. But, let's start our discussion with x where the voting procedure requires a majority vote.

In fact, proposal x is *not* stable. To understand why this is so, notice from Fig. 3b that voter one would prefer any proposal which is closer to his ideal point. That is, this voter would prefer anything inside the circle where the center is his ideal point and the circle passes through x. What Fig. 3b indicates, then are the reactions of the three voters; each of them prefers any proposal inside of the circle centered at their ideal point and passing through x. The instability can be explained by use of high-school geometry; any two of these discs have a non-empty intersection. Consequently, any proposal inside one of the three Fig. 3b lenses will beat proposal x with a majority vote.

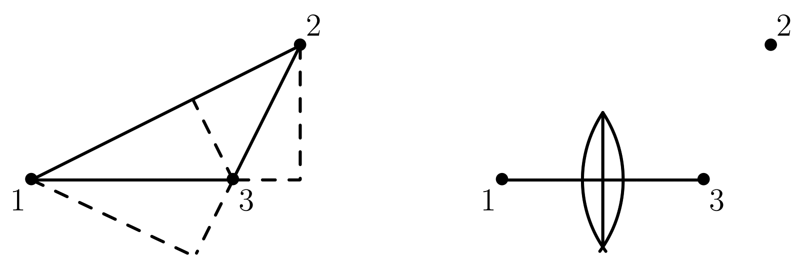
With reflection, it becomes clear that we all have experienced this phenomenon. No matter how fair and carefully crafted a proposal, it is possible that some group will make changes. And, since this is done by a majority vote, we leave a meeting with the comments that the proposal has been "improved." Has it?

The point is that with these ideal points, it is impossible to have a proposal which is not subject to some "tinkering." This is because any positioning of point x will allow a similar intersection affect where some majority can force a change. And then the changed proposal is open for more adjustments. And that proposal ... For veterans of departmental meetings, does this process sound familiar?

3.3. **Chaos theorem.** The following remarkable theorem found by Richard McKelvey [23, 24] shows that this "adjustment" process can be highly chaotic. Stated in terms of Fig. 3a, his result asserts that you can choose *any* two proposals $\mathbf{X}^1$ and $\mathbf{X}^2$; it does not matter where they are located. McKelvey proved that there exists an agenda — a listing of proposals $\{\mathbf{x}_j\}_{j=1}^N$ where $\mathbf{x}_1 = \mathbf{X}^1$ and $\mathbf{x}_N = \mathbf{X}^2$ — where $\mathbf{x}_{j+1}$ beats

$\mathbf{x}_j$, $j = 1, \ldots, N-1$, in a majority vote. In other words, not matter what the initial proposal, it is possible to direct the discussion so that by majority votes, the outcome ends at any other specified outcome; this is true no matter what is the specified final outcome. So, it is possible to start with the x on Fig. 3a, and by majority votes and haggling of the voters, end up at a proposal extremely far to the right of this page. McKelvey's result is based on arguments from differential topology.

McKelvey's result is called the "chaos" theorem for good reason. By selecting $\{\mathbf{X}^j\}$ in various positions, McKelvey's theorem states that the agenda process can enter almost any region and then depart to go somewhere else. (As one should expect, McKelvey's result attracted considerable attention — and alternative proofs. An interesting one coming the closest to capturing the flavor of "chaotic dynamics" was developed by D. Richards [33].)



a. Three altitudes **b.** Symmetries

Fig. 4. M. Tataru's analysis

McKelvey proved the situation can be bad, but, how "bad can it be?" This question was addressed by my former Ph. D. student Maria Tataru. One of the issues which interested her was to find the minimal value of N; for instance, if N is large, such as 10^{100}, then the conclusion is not practical if only because no departmental meeting will last long enough to introduce all options to move from a specified starting location to a desired ending one. To introduce her result in terms of three voters, consider the diagram of Fig. 4a showing three ideal points and the triangle they define.

Three of the dashed lines in Fig. 4a define the three altitudes of the triangle; let c be the minimum altitude. In this setting Tataru [71, 72] established both upper and lower bounds on the value of N; in particular, she showed that

$$\frac{2||\mathbf{X}^1 - \mathbf{X}^2||}{3c} \leq N \leq \frac{||\mathbf{X}^1 - \mathbf{X}^2||}{c} \tag{3.2}$$

where $||\mathbf{X}^1 - \mathbf{X}^2||$ is the usual Euclidean distance between the specified beginning and ending proposals.

This result means, then, that an equilateral triangle configuration of ideal points leads to the worse possible outcomes. On the other hand, the flatter the triangle, the smaller the c value, hence we have a larger N value. In turn, the larger N value makes it more unlikely that there ever will be time enough to carry out the agenda process of making it to $\mathbf{X}^2$. This is a very nice result.

In the process of obtaining her conclusion, she reproved McKelvey's result by using a different approach. The basic insight is indicated in Fig. 4b where we can think of the current proposal considered by these three voters as located at the top of the lens. Her argument exploits the symmetry required by any lens; notice that the lens has symmetry with respect to two axis. While it is an oversimplification, one can view Tataru's result as finding the algebraic orbit of any of these sets under the induced symmetry groups.

3.4. **Stability and the core.** Now that we know it is possible to have "chaotic behavior" in voting, the next issue is to search for stability; can it exist? In fact, it can. The above geometric argument, for instance, shows that whenever the three ideal points define a triangle, chaos reigns. But, this argument fails should the three ideal points lie along a line; here the circle argument need not provide opportunities for any majority coalition. The goal is to make all of this mathematically precise.

Definition 1. *Point* **p** *is called a core point if there does not exist any other point which can beat* **p**. *The set of all core points is called the core.*

Examples are immediate. For instance, Fig. 5 displays two situations; in Fig. 5a the three ideal points are along a line; in Fig. 5b there are four ideal points along the line. The question is to determine whether there exists a core for these two configurations and to find it.

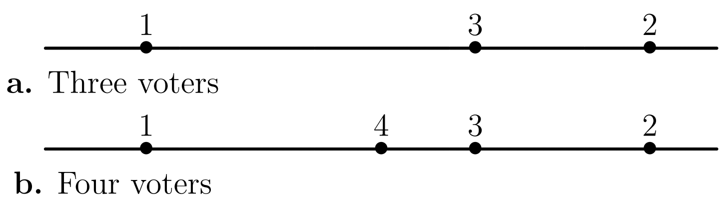


Fig. 5. Two examples of cores

It is quite clear that the core for Fig. 5a consists of voter 3's ideal point, and it consists of this point only. This is because for any point selected to the left of this point, a majority consisting of voters 2 and 3 prefer voter three's ideal point. Similarly, for any proposal to the right of this point, a majority coalition consisting of voters 1 and 3 prefer voter's three ideal point. Hence, voter three's ideal point is the only core point. A similar analysis demonstrates that the core for the four voter setting of Fig. 5b consists of the closed interval defined by voters 3 and 4 ideal points.

This result is known in the literature as the *median voter theorem* and it goes back to the work of Hotelling in the 1920s. This conclusion has been used as an explanation for some of the blandness occasionally found in political parties whereby to win, the parties try to be positioned *not in the middle* but near the views of the median voter. Indeed, Fig. 5a indicates a situation where rather than at the middle, the median voter has leanings to the right.

3.5. **Extensions.** Extensions of this modelling have been used to obtain insights into a wide variety of political behavior. Quite frankly, right now this approach is a "growth industry" in mathematical political science which has provided many fascinating assertions. But, I want to move in a particular direction which leads to some interesting mathematics of the kind one might be surprised to see associated with the mathematical social sciences. But first let me introduce terminology which allows generalizations.

Beyond majority voting, there are supermajority methods where a winning proposition must receive at least a specified number of votes which could be more than 50%. An example described in Saari [44] is the election of a pope; to avoid problems which caused grave difficulties for the Catholic Church, a pope must receive over two-thirds of the vote. There are procedures in the US where a successful vote must have at least 60% support, or even a 75% support.

Definition 2. *With $n \geq 3$ voters, a q-rule is where a winning proposition must receive q or more votes. If $q = n$, this is called the unanimity rule; if q is the first integer greater than $\frac{n}{2}$, this is the majority rule.*

The question is to determine whether the above core and chaos results extend to q-rules. As indicated below, they do.

The next extension is to move toward more realistic modelling by generalizing the "Euclidean preferences" where preferences are given by Euclidean distances. After all, some voters might put more weight on one issue rather than other, and this could cause the preferred proposals to be in the interior of an ellipse rather than a circle. Even more general, a curve expressing indifference may be based on trade-offs so the shape of the curve could change from location to location.

Definition 3. *A utility function is a smooth function from issue space to the real line. The level sets indicate proposals which the voter treats as indifferent, or has the same preference for them. Proposals with larger values of the utility function are more preferred by the voter. An "Euclidean preference" is where the utility function is given by the negative of the the Euclidean distance of a proposal to the voter's ideal point.*

The "smoothness" is imposed for mathematical convenience. Similarly, I will require the utility functions to be strictly convex. To explain what I mean, at each point the gradient defines a hyperplane; locally the level set passing through the point is strictly on the same side of the hyperplane. By extending the space of functions from Euclidean distances to an open set of continuous functions, we should expect a wider assortment of results, and this happens.

3.6. **Stability and instability coexisting.** As a person whose research interests also include the Newtonian N-body problem, a setting where chaos was first discovered by H. Poincaré and where we now know that instability and stability can coexist, I find that the above provides an interesting and somewhat similar picture. But first, by using the above terms, a more precise version of what McKelvey and Tataru proved can be stated.

Theorem 3. *(McKelvey [23, 24].) If the ideal points create a setting where the core for a majority vote is empty, then for any two points, $\mathbf{X}^1$ and $\mathbf{X}^2$, it is possible to find a listing of proposals $\{\mathbf{x}_j\}_1^N$ so that $\mathbf{x}_1 = \mathbf{X}^1$, $\mathbf{x}_N = \mathbf{X}^2$ and $\mathbf{x}_{j+1}$ will beat $\mathbf{x}_j$, $j = 1, \ldots, N-1$, in a majority vote.*

Tataru extended this theorem in two ways.

Theorem 4. *(Tataru [71, 72] The above theorem holds for any q rule. Moreover, an upper and lower bound on N depends on the Euclidean distance $||\mathbf{X}^1-\mathbf{X}^2||$ and multiples based on how far the ideal points are from defining a core, where the closer the ideal points are to defining a core, the larger the value of N.*

Thus, for all of the q rules, we have a combination of chaos and regularity where the effects of chaos diminish as the stable setting is approached. Incidentally, when stability is given because of a core, this core is an attractor in the sense that each proposal not in the core can be beaten by a proposal which is closer to the core.

3.7. **Structural instability.** The story is not over; it turns out that this delicate balance between stability and instability can be very delicate. To illustrate, the configuration of ideal points in Fig. 6 allows voter seven's ideal point to be the core. The argument is the same as the one given above for points along a line. If any other proposal is selected, then the vector from this proposal to voter seven's ideal point defines a vector, and this defines an orthogonal line passing through voter seven's ideal point. Voter seven and all voters on the other side of this dividing line prefer voter seven's ideal point to the new proposal. Hence, this ideal point is unbeatable, so it is the core.

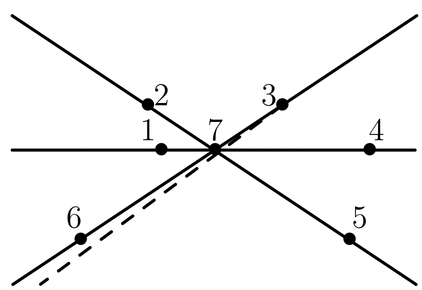


Fig. 6. A structurally unstable core

The existence of a core in this setting is established by the incredible symmetry in the location of the ideal points as established by this "Plott diagram" which was developed by C. Plott [32]. Now, break this symmetry by slightly moving any of the first six points off of the lines of symmetry; e.g., slightly move voter six's ideal point to the right so that it no longer is on the solid line but on the dashed line connecting the ideal points for voters three and six.

This slight change kills the core. To see this, find the point on the dashed line which is closest to voter seven's ideal point. Although this point is very close to voter seven's ideal point, since the new point defines a direction orthogonal to the dashed line, it follows from elementary geometry that this change creates a proposal which is closer — hence more preferred — for voters three, four, five, and six which constitute a majority. A very slight change you might argue. But, just like making a slight change in the position of a pendulum standing upright, this change unleashes the instability of the McKelvey theorem which is described above.

This example illustrating the fine, delicate line between the stability of a core and the instability when a core is empty leads to another question which seems to have been first explored by McKelvey and them McKelvey and Scholfield [25, 26, 57]. The question is to find the dimension of issue space where very slight changes in the locations of the voter ideal points will preserve the core. There were several attempts at finding the answer, but Banks [5] found errors in the published accounts; errors both in the proofs and in the assertions. The final answer is given in Saari [43] for utility functions, Euclidean preferences, and any q rule. Rather than giving the full result, I want to emphasize the mathematics.

3.8. **Using singularity theory.** Suppose we wish to check whether a point $\mathbf{p}$ is a core point. By using utility functions, rather than Euclidean distance, all interest shifts to the gradient of each utility function located at $\mathbf{p}$; that is, we are interested in the relationships admitted by the set $\{\nabla u_j(\mathbf{p})\}_{j=1}^n$. But, the full gradient is not of interest; we only are interested in the direction of the gradient. After all, this direction indicates the direction of personal interest for each voter.

If $\nabla u_j(\mathbf{p} = \mathbf{0}$, then we are at the ideal point of a voter. It is easy to show that it is generically unlikely for two voters to have the same ideal point (i.e., a very slight change in their preferences will separate the ideal points), so assume that at $\mathbf{p}$ at most one voter has the gradient condition $\nabla u_j(\mathbf{p} = \mathbf{0}$. As for the other voters, since the interest is centered on the directions

$$\{\frac{\nabla u_j(\mathbf{p}}{||\nabla u_j(\mathbf{p}||}\}_{j=1}^n,$$

the argument can be identified with points on a sphere S^{k-1} where k is the dimension of issue space. This is illustrated in Fig. 7 where $\nabla u_1(\mathbf{p})$ is replaced by the bullet.

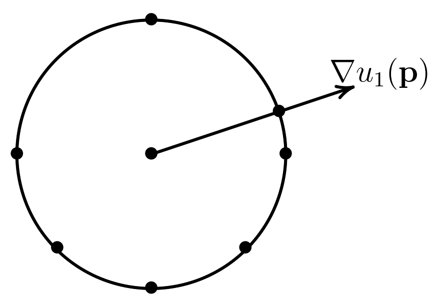


Fig. 7. Gradients to points on unit ball

To repose the problem in this geometric setting, it appears (and it will turn out to be true) that for each q rule, the problem can be described in the following manner.

> For any n, find the largest dimensional sphere, S^{k-1} where it is possible to position n points on the sphere which satisfy the following condition. For all possible ways to pass a hyperplane through the center of the sphere, at most $(q-1)$ points are on either side of the hyperplane. Moreover, this condition holds for any slight changes in the locations of the points.

If the core involves the ideal point of a voter, then the mathematical problem becomes the following:

> For any n, find the largest dimensional sphere, S^{k-1} so that it is possible to position $(n-1)$ points on the sphere which satisfy the following condition. For all possible ways to pass a hyperplane through the center of the sphere, at most $(q-1)$ points are on either side of the hyperplane. Moreover, this condition holds for any slight changes in the locations of the points.

Showing that it is possible to find the positioning of the points to satisfy this condition is easier than showing that the condition is robust. Verification of this robustness condition comes from singularity theory.

While there are many excellent books on this singularity topic, my favorite remains one of the earlier ones; it is the book by Golubitsky and Guillemin [12]. Since details can be found in this reference, I will provide only an intuitive argument.

3.9. **Inverse function theorem and Jet spaces.** The argument starts with the inverse function theorem. We know that for a smooth mapping

$$F : R^n \to R^m$$

and $\mathbf{p} \in R^m$, then in general and locally $F^{-1}(\mathbf{p})$ is a $n-m$ dimensional manifold. So for $n = m$, expect $F^{-1}(\mathbf{p})$ to consist of isolated points. If $n > m$, expect $F^{-1}(\mathbf{p})$ to consist of $n-m$ dimensional manifolds. But if $n < m$, expect, in general, that $F^{-1}(\mathbf{p})$ is empty. The usual condition ensuring a manifold of the correct dimension, of course, is a rank condition imposed on the differential DF.

Instead of the inverse image of a point, suppose our interest is in the inverse image of a smooth manifold $\Sigma \subset R^m$ of dimension σ. Here, the rank condition is replaced by a transversality condition. Namely, at a point $\mathbf{p} \in \Sigma$, the linear space spanned by the tangent space $T_{\mathbf{p}}\Sigma$ and the plane defined by $DF(R^n)$ must have the full dimension m. In this case, along with the usual caveat that the results are local, the dimensionality of $F^{-1}(\Sigma)$ is $n - [m - \sigma]$. Namely, the codimension of Σ defines the codimension of $F^{-1}(\Sigma)$. (For intuition about this result, suppose that Σ is given, locally, by $g^{-1}(0)$ where $g : R^m \to R^{m-\sigma}$. This means that $F^{-1}(\Sigma) = (g \circ F)^{-1}(0)$. Now apply the usual rank condition from the inverse function theorem to $Dg \circ DF$.)

This basic structure is applied to a space which includes derivatives, etc. So, when considering mapping $f : R^n \to R^m$, let J^1, the associated jet space with first order derivatives, be

$$J^1 = R^n \times R^m \times [R^n \times R^m].$$

Think of this as consisting of all possible domain points, all possible images, and all possible choices for the derivatives Df.

A given map f defines a mapping

$$j^1(f) : R^n \to J^1$$

in the following natural manner:

$$j^1(f)(x) = (x, f(x), Df(x)).$$

Conditions we wish to impose on the function define manifolds $\Sigma \subset J^1$. To illustrate, suppose we are interested in the critical points of functions $f : R^2 \to R^1$,. Here,

$J^1 = ((x, y); z; (A, B))$ where $A, B \in R^1$. As a critical point is where the partials are zero, we have that

$$\Sigma = ((x, y); z; (0, 0)).$$

Thus, for any f, the locations of the critical points are given by

$$[j^1(f)]^{-1}(\Sigma).$$

The point to notice is that Σ has codimension two; this is because two of the variables are specified. Consequently, we should expect, in general, that the critical points form manifolds of dimension $2 - 2 = 0$; i.e., expect the critical points to be isolated.

Now suppose we wish to find the critical points where $f(x, y) = 1$. For this problem, the jet space condition is

$$\Sigma_1 = ((x, y); 1; (0, 0)).$$

Here, Σ_1 has codimension three, so $[j^1(f)]^{-1}(\Sigma_1)$ should have codimension three, or dimension $2-3 = -1$. Since it is impossible to have a negative dimension, this assertion means that in general this condition cannot be expected to be satisfied. Yes, it is easy to construct examples with this property, but the relevance for our discussion is that, just as with the Plott diagram of Fig. 6, a slight change in the function violates the desired condition.

As an example of mixed conditions, suppose the goal is to study the structure of points where the gradient and the vector defined by (x, y) are orthogonal. Here

$$\Sigma_2 = \{((x, y); z; (A, B) \mid xA + yB = 0\}.$$

This single constraint means that Σ_2 has codimension one. Consequently, in general, this condition $[j^1(f)]^{-1}(\Sigma_2)$ can be expected to be (locally) satisfied along some collection of curves in R^2.

So far we have been appealing to our intuition developed from the inverse function theorem. To be precise for any given f, we would need to verify the transversality condition. But this is a mess; the computation of $D(j^1(f))$ for the first two factors is only the identity mapping, but it involves D^2f for the last factor. But, if we are interested only in generic conclusions, then we are saved by an important result obtained by R. Thom (e.g., see Thom [75]).

By imposing an appropriate topology on the functions in function space, known as the Whitney Topology, Thom proved for these jet mappings that, generically, either the mapping misses the target Σ or it meets it transversely. This means that once Σ is defined, if it can be established that some mapping f allows its jet map to be in Σ, then generically the jet map meets transversely. Consequently, all of the above codimension comments are established generically.

The interesting fact about this mathematical structure of jet mappings and singularity theory is that it can be applied to analyze a wide range of questions coming from the social sciences. For instance, the economists have a concept call the "Pareto equilibrium;" a concept introduced by a nineteenth century engineer who became one of the first mathematical economists. A Pareto point is where any change, any reallocation of the resources among the players, will hurt someone.

Smale [64, 65] established a derivative condition on the gradients of the utility functions to characterize these Pareto points. Using Smale's characterization, Saari and

Simon [52] developed an appropriate singularity theory[3] to establish the mathematical structure of these sets; in general, the Pareto points form submanifolds of an appropriate dimension.

3.10. **Back to the generic stability of the core.** By using this kind of singularity argument I established the connection between the stability of the core, the choice of the q-rule, and the dimensions of issue space. Part of the challenge comes from the above step in the description of singularity theory which requires showing that *some* collection of utility functions will satisfy the positioning condition of points on the sphere as described above. Recall, this is equivalent to showing that points on S^{k-1} can be positioned in an appropriate manner; clearly we are searching for a symmetric positioning. For S^1, all of this is trivial; just place the points symmetrically along the circle. But, problems already arise with S^2. Four points can be placed at the vertices of a equilateral tetrahedron. But, what is the symmetric positioning of five points on S^2? Six points? Seven points? How about the symmetric positioning of seven, or eight, or ... points on S^3, or S^4, or ... ?

In many cases, an answer can be found by extending the Plott diagrams in a natural manner into the different dimensional settings. But to handle other cases, I called on my background in the Newtonian N-body problem. Let me briefly explain.

One issue for the N-body is to find what are called *central configurations.* These are the configurations where there is a common scalar λ so that the position of each particle is precisely λ times the acceleration imposed on the particle. The importance of these configurations is that they identify limiting configurations of collisions, of expansions, and other Newtonian properties.

A special case of these central configurations, when all of the masses are equal, involves the positioning of the points on a sphere. Now, in the N body problem, physical considerations restrict attention to S^1 or S^2. But, the same types of technicalities involving the dynamics hold for any k, and it establishes that for any S^{k-1} some positioning does what we want. The idea is to let "gravity" move the particles around. For instance, if a plane passing through the center of a S^k divides the points so more are on one side than the other, then gravitational forces attract the particles to a more balance setting. The robustness comes from singularity theory.

It turns out that the Σ conditions which arise for different dimensions of issue space and different choices of n become quite complicated. The reason is based on the intricacies which are allowed by various dimensional spaces. Since the complicated versions of Σ give rise to complicated sounding conclusions, I leave the precise statement of what can happen to Saari [43]. However, one situation provides a particularly intuitive interpretation, and this situation is described next.

[3] A change in the singularity theory was required because the standard theory required each person's utility function to depend upon what each and every other agent would receive. Thus, the standard assumptions which are imposed by the economists would relegate the modelling to "generically unlikely" settings. By accepting the mathematical challenge of creating a singularity theory peculiar to the economists assumptions, results could be found. This, again, illustrates how questions from the social sciences creates new mathematical directions.

One case is that the dimension of issue space which allow a stable core is bounded by the number of people it takes to change their mind to change the outcome. Let me illustrate with examples.

Consider a majority vote with an odd number of voters, say $n = 11$. Here, a proposal can win with six votes. But, it takes only one voter to shift the balance; if one voter voting for the winning proposal changes his mind to vote for the other proposal, then it will be the new winner. This means that

> *for the majority vote with an odd number of voters, the core is structurally stable only in a one dimensional issue space.*

Now consider the majority vote with an even number of voters, say $n = 12$. Here, a proposal needs seven votes to win. Consequently, it takes *two* voters to change their minds to allow the previous loser to become the new winner. This means that

> *for the majority vote with an even number of voters, the core is structurally stable in a one or two dimensional issue space.*

As a different example, with seven voters, consider the $q = 5$ rule. Again, this means that a winning proposal needs five votes; to change the outcome, three of these voters have to change sides to join with the other two to have a winning coalition of five voters. Consequently,

> *with seven voters and $q = 5$, we can expect a core to be structurally stable in a one, two, or three dimensional issue space.*

As an illustration of this $q = 5$ rule, notice that in Fig. 6, voter seven's ideal point *remains a core point* with slight alternations of six's ideal point, or any slight change in any of the seven voters' ideal points. This sure differs from the instability of this configuration for the majority vote.

Based on this analysis, we now can appreciate the stability provided to the Catholic Church by their change in the way a pope is elected. For sake of argument, suppose there are 100 voters. With the previous "majority vote" approach, an outcome can be altered by only two voters changing their minds. Consequently, according to the theorem, the setting would remain stable with only two basic issues on the table. With the two-thirds requirement, a winning candidate requires 67 votes. Thus, it would take 34 voters to change their mind in order to alter the conclusion. Consequently, according to the mathematics of singularity theory, expect this solution to remain stable for up to 34 different issues.

Do politicians know about this delicate balance between the number of issues and the stability of outcomes? While it is quite reasonable to expect that they do not know the mathematics, it is equally as reasonable to expect that they most surely understand what is going on in a pragmatic sense. As proof, consider what happens during a close election where one candidate is winning. The losing candidate has do something to change the election attitude; to have a chance at winning, this candidate has to impose instability. According to the above mathematical description, this instability can be done by changing the dimension of issue space — by introducing new issues. We sure see this behavior in action during any election season! Often this effect is manifested in terms of negative ads, but whatever is done, it is a part of our political reality. On the other hand, who would have thought that the onslaught of name-calling and finger pointing TV political advertisements is a manifestation of mathematical singularity theory?

To end this section, it is worth pointing out that the basic message and theme of these notes remains intact; the social sciences can be a source of a large number of fascinating mathematical structures. Indeed, the above addresses the question of the stability of a core with *one class of decision rules*, the q-rules. There are many other issues coming from cooperative games and so forth which lead to more challenging singularity theory questions.

4. Evolutionary game theory[4]

So far, I have shown how addressing natural issues coming from the social sciences leads to mathematical questions of symmetry, extensions of techniques coming from chaos, and even singularity theory. This section will show how other concerns are naturally addressed by using dynamical systems and even index theorems. Again, the concepts are introduced in terms of the social science issues.

4.1. **Game Theory.**

Let me start with game theory. A good reference providing an introduction to this topic is the book by Aliprantis and Chakrabarti [1]; indeed, while still at Northwestern University, I introduced this book to be used with our undergraduate program in the mathematics department and they still are using it.

For intuition, first think of a game of checkers or chess. Your object is to win; it is to optimize. However, the problem that you must optimize is determined by the actions of someone else — the play she just made. Similarly, what your opponent is forced to optimize is determined by your actions.

So, think of game theory as an extension of the usual mathematics of optimization where now several people are optimizing potentially different functions and where the precise function each person must optimize depends upon the actions of others.

4.1.1. *Some simple examples.* Wow. What a mouthful. Let me start with a simple example involving a common stratagem in detective TV shows; the police apprehend two people involved in a misdemeanor primarily because they believe they are guilty of a far more serious crime. You know the plot; the police separate the suspects into separate rooms where they play "Good cop; bad cop." Each suspect is told that if he squeals on his partner, he will be set free, but his partner will spend the next 10 years in jail. If they both confess, then they probably will each spend five years in jail. But, if the partners in crime cooperate by not confessing anything, they will each spend a half year in jail for the misdemeanor.

	C	**D**
C	$(-.5, -.5)$	$(-10, 0)$
D	$(0, -10)$	$(-5, -5)$

(4.1)

All of this information is posted in the Eq. 4.1 table where the row and column data are, respectively, suspect 1's and 2's options. Here C means "cooperate" – not with the police but with his partner by staying quiet – and D means to defect by telling on

[4]The results at the end of this section have been reported in numerous seminars and colloquia; some of them are detailed in Saari [49].

the partner. So, the (C, D) entry of $(-10, 0)$ reflects the possible outcome of the above story where suspect one will go away for ten years while suspect two is released.

The optimization problem follows the script of the TV show; suspect one worries about what his partner will do. If his partner keeps quiet, suspect one is maximizing over the first entries in the first column; here it is in his best interest to "Defect" to avoid spending a half year in jail for the misdemeanor. But if his partner defects, then the first suspect must optimize over the first entries in the second column; here he has to choose between spending ten years in jail vs. only five. Again, his best choice is to "Defect." The symmetry requires the same analysis to hold for the second suspect. Thus, the conclusion of the "Prisoner Dilemma" game is for each to confess and, by making bad decisions for the two of them, rid the streets of crime.

Rather than an isolated example, this Prisoner Dilemma game captures the essence of a reasonable portion of a conflict between personal advantage and the optimal setting for a group. Indeed, this phenomenon occurs whenever

$$\begin{array}{c|c|c} & \mathbf{C} & \mathbf{D} \\ \hline \mathbf{C} & (a_1, a_2) & (b_1, c_2) \\ \hline \mathbf{D} & (c_1, b_2) & (d_1, d_2) \end{array} \quad \text{where } c_j > a_j > d_j > b_j,\ j = 1, 2. \tag{4.2}$$

As a result, this kind of game captures all sorts of daily frustrations such as the familiar kind which occurs during construction season when the sign very clearly asks drivers to merge to the left. If everyone would cooperate by merging when requested, there would be a smooth transition. But, those who defect by staying in the right lane until the last instance achieve a personal gain of a few minutes of driving time at the cost of penalizing the cooperators by creating a traffic jam.

Another example involves neighboring communities facing the problem of the homeless. If both cooperate by providing food, shelter, training, and so forth, each community incurs a cost which is tempered by positive effects of addressing the problem. But a defecting community saves the expense of the program and the homeless probably will leave the defecting community. After all, the homeless still need food and shelter, so they will migrate to the "cooperating community." In turn, not only does this migration increase the cooperating community's costs, but it will generate all of the problems associated with having a huge influx of homeless. If both communities defect, the homeless problem continues without solution.

This is not an hypothetical exercise; a scenario of this sort caused Evanston, Illinois, to suffer the problems which accompany a sudden increase in the homeless population; indeed, the problems extended to where, for a period, so many homeless used the public library for warmth and the rest rooms that regular library patrons found it to be a difficult situation. Indeed, change "homeless" to "addicted to drugs," and a well documented international example is Zurich. Recall, because Zurich tried to address the health problem with clean needles and other programs, but because other neighboring countries did not, Zurich quickly became a magnet for the addicted.

4.1.2. *Other settings; many extensions.* The main point examples is that even simple game theory models can provide valued insight into problems of the social sciences. One should expect different versions of games to arise whenever different individuals try optimize different objectives. This includes a department where faculty members want to optimize certain objectives while the chair has a different set; it occurs with the department chairs trying to optimize one set of objectives while the dean has another. It occurs in industry where the design engineers, the manufacturing segment, and the marketing group optimize slightly different objective functions.

Game field has progressed beyond specified games to understand how to alter games. As illustrations, a large component of economics is based on "incentives" — the design of economic settings so that the "optimal strategy" for individuals is to preform as desired. For instance, a version of the "Principal-agent" problem has the person who owns a firm hiring someone else to run it. How should the contract be designed to ensure positive performance? In another direction, Sieberg [60] models how certain changes in the Prisoner Dilemma game Eq. 4.2 — change different groups impose on the situation to change the game to their advantage — provides insight into a variety of social science phenomenon such as violence in gangs, pimps and prostitution, drugs, and so forth.

Game theory, in other words, rapidly is becoming a central force in the social sciences. Of interest to mathematicians is that this field, which remains in an early stage of development, poses fascinating mathematical structures which are waiting to be addressed. To identify some, just think any setting considered in optimization; function spaces, stochastic processes, uncertainty, and on and on. Each optimization setting can be naturally generalized into a more complex game theoretic concern. What I describe next is a role for dynamical systems and even some algebraic topology.

4.2. **Dynamical models.** Dynamical game theoretic models with varying degrees of mathematical sophistication have been around for decades. As just one of many illustrations, let me point to Avner Friedman's book [11] on this topic. But right now I will use an easier representation.

4.2.1. *PC vs. Mac.* Although there are many who consider Apple's products superior to the PC world, the PC dominates. Why is this?

Part of the problem reflects what economists call "externalities;" in our simple illustration, these are the added benefits, or losses, one experiences by others having either a PC or a Mac. For instance, if more people have Mac's then I will benefit by the wider availability of software, or by having colleagues who can help me set up my system or resolve those ever present computer problems. Indeed, many of us probably have experienced this phenomenon by adopting a particular kind of computer, rather than one we might prefer, because the adopted brand is dominant in the department.

To model this conflict, let the two brands be identified by X and Y and let x and y, satisfying $x + y = 1$, describe the market portion of each brand. The constraint normalizes the setting to indicate the market share with the unit interval $0 \leq x \leq 1$ where the $x = 1$, $y = 0$ endpoint, the left-hand endpoint, means that brand X fully dominates the market, and $x = 0$, $y = 1$, the right-hand endpoint, means that brand Y dominates.

Economists (and I regret that I do not have the reference) have modelled how the current level of market share influences new purchases. Rather than specifying the actual equation, the solution is indicated in Fig. 8a. The important point to notice is that there are three equilibrium points; two are at the endpoints where each is an attractor indicating that once a product sufficiently dominates, it gains full control of the market. The remaining interior equilibrium is a repellor; everything moves away from this point. What adds interest to this equilibrium is that it indicates the existence of a threshold. Namely, according to this simple model, the only way one brand or the other will dominate and even drive the other out of business is to have a sufficiently large share of the market so that the externalities will continue the market movement.

a. Market competition **b.** Ultimatum game

Fig. 8. Simple dynamical models

This dynamic of Fig. 8a, then, can translate into policy decisions. For instance, it is not unusual to see companies at an early stage provide high incentives for people to use their product. In fact, recall the donated computers to schools, libraries, etc. This charitable act, in fact, translates into excellent business practice by allowing the brand to cross the "threshold."

4.2.2. *Ultimatum games.* Another example where dynamics has provided insight into the interactions is the so-called Ultimatum game. The rules of the game, which follow, captures the spirit of several business "take it or leave it" economic settings.

> Two players are to split a sizable amount of money; say $100,000. One person, say you, has to decide how much of $100,000 to offer the other player; denote this amount by $x. If the other player accepts the offer, she has $x and you have $(100,000 - x). There are no negotiations; only the offer can be made and then accepted or rejected. How much should you offer?

The standard response in most western countries is to offer a 50–50 split. Indeed, with many experiments involving serious amounts of money, it has been shown over and over again that an offer much less than 50% will be rejected. The question is to understand why this is so. After all, out of the $100,000, why not offer $30,000 rather than $50,000? By rejecting that amount, the other person has nothing, so she has an incentive to accept. Continuing, why not offer only $1,000? Why not just $1? Yet, that 50-50 split dominates so firmly that it can be treated as a social contract.

An approach explored by Brian Skyrms [62, 63] describes this decision process in terms of a social contract which has evolved over time. Details can be found in his book, but let me indicate part of his argument which uses a simplified setting where there are two kinds of people; those who demand $\frac{2}{3}$, and those who want only $\frac{1}{3}$.

If people of the $\frac{2}{3}$ persuasion meet, they will leave with nothing. We can capture this as stating that "in the next generation, they will not replicate." On the other hand, two $\frac{1}{3}$ kinds will reach an agreement, but as they do not fully utilize all of the resources. In the next generation they will replicate but not as fully as they could have if they had used all of the available resources. Of course, if a $\frac{2}{3}$ and a $\frac{1}{3}$ type meet, an agreement can be made and both can replicate according to the amount of resources they consume.

The “replicator dynamics” captures this growth sense where individuals meet at random. It is not difficult to write down equations capturing this spirit. To keep the solution on the line segment $x + y = 1$, where the left hand endpoint of $x = 1$ means that only the $\frac{2}{3}$ types exist, obvious normalizations are made. The actual equations coming from Skyrms [63] are

$$x' = \frac{x}{3}[2y - (x + 2xy)], \quad y' = \frac{y}{3}[1 - (x + 2xy)] \tag{4.3}$$

but the precise form is not needed for what follows.

The solution of Eq. 4.3 is illustrated in Fig. 8b. Again, a point on this line segment illustrates the portion of the full population occupied by each type. It is interesting to notice how portions of the solution capture intuition. For instance, the $\frac{2}{3}$ end of the segment identifies a population where almost everyone is greedy. But, greedy people interacting with greedy people means no agreement will be struck, so they will tend to die out causing the dynamic to tend to the right. The $\frac{1}{3}$ end, on the other hand, provides a heavenly opportunity for the greedy people. Here, almost everyone they encounter with the random selection process will give them plenty of resources, and they will prosper – forcing the dynamic to the left.

The interesting feature of this solution is the equilibrium point — an attractor — indicating a balance of the two kinds of individuals. This equilibrium indicates a stable balance between two kinds of people – the greedy and the generous – whereby slight increase in one species assists the growth of the other species.

4.2.3. *Toward a 50-50 division.* But, where does that 50-50 division described above come about? Skyrms studies this by using the dynamics to determine the consequences of introducing a mutant in the $\frac{2}{3} - \frac{1}{3}$ world who asks for precisely half.

Here the replicator dynamics is designed precisely as above except it has to account for a random interaction of a $\frac{1}{2}$ type with the other types. Again, if a $\frac{2}{3}$ and a $\frac{1}{2}$ interact, no deal is struck; if a $\frac{1}{3}$ and a $\frac{1}{2}$ meet, a deal is struck but the $\frac{1}{2}$ type gains strength with the added resource, and a meeting of two $\frac{1}{2}$ types leads to a division and growth. After normalizing the obvious equations so that the solution remains on the simplex $x + y + z = 1$, the equations become

$$\begin{aligned} x' &= x[\tfrac{2y}{3} - (\tfrac{x}{3} + \tfrac{2xy}{3} + \tfrac{1-x}{3})], \\ y' &= y[\tfrac{1}{3} - (\tfrac{x}{3} + \tfrac{2xy}{3} + \tfrac{1-x}{3})], \\ z' &= z[\tfrac{(1-y)}{2} - (\tfrac{x}{3} + \tfrac{2xy}{3} + \tfrac{1-x}{3})] \end{aligned} \tag{4.4}$$

where the general solution has the form depicted in Fig. 9.

Notice the several interesting features which emerge from Fig. 9. The first observation is that adding the new type of player forces at least five equilibria points. The three along the bottom edge are the original ones depicted in Fig. 8b. One of the two new ones is an attractor located at the top vertex of the triangle. Indeed, solutions along either of the two edges must move toward this point; this indicates the superior power of the $\frac{1}{2}$ type which drives the other two types out of existence.

The other new equilibrium point is in the interior of the triangle; it has a “stable” and “unstable” manifold. Recall, a point along the stable manifold must progress directly to the equilibrium point. A point on the unstable manifold, indicated by the arrows pointing away from the equilibrium, moves away in the indicated direction. For a point near, but not on the stable manifold, the solution tends toward the interior equilibrium until it gets nearby. Then the effects of the unstable equilibrium take over to force the solution away from the interior equilibrium in the indicated direction.

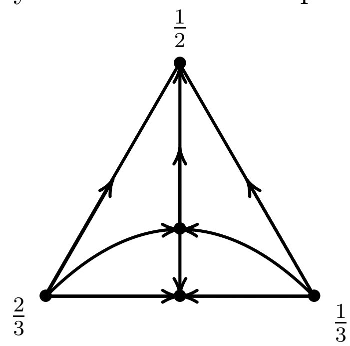


Fig. 9. Ultimatum with three types

Several interesting implications come from this dynamic. For instance, the dynamics indicate that there are situations where the $\frac{1}{2}$ mutant can be driven to extinction. More precisely, the solution for any initial point below the two arcs serving as the stable manifold for the interior equilibrium tends to the equilibrium on the bottom edge. Recall, at this equilibrium, there are an equal proportion of $\frac{2}{3}$ and $\frac{1}{3}$ types but no $\frac{1}{2}$ types. In other words, similar to the "PC-Mac" example, a threshold is imposed by the arcs consisting of the stable manifolds of the interior equilibrium point; only if there are enough of this half type to bridge this threshold can the $\frac{1}{2}$ types survive.

And survive they do! With a sufficient number of $\frac{1}{2}$ types, all solutions tend toward the attracting equilibrium located at the top vertex of the triangle; at this equilibrium, only the $\frac{1}{2}$ types survive. A social custom, or social contract, has been established whereby people wanting less than $\frac{1}{2}$ recognize they can gain more; those wishing more than $\frac{1}{2}$ are rebuffed. The interesting fact is that this message suggested by a particularly simple dynamical model has been confirmed over and over again with experiments.

4.3. **What next?** One can imagine the wide variety of possible extensions. For instance, in the Ultimatum game, what happens if an individual demands a certain amount when making the offer, but demands another amount when required to either accept or reject the offer?[5] All sorts of extensions can be imagined which require dynamical setting with surprisingly large dimensions. It is easy to imagine the challenging and interesting mathematics associated with these models. For instance, since these questions can easily involve a dynamical system in a 70, or a 90 dimensional space, we are discussing systems where it is not easy to determine the equilibria along with their hyperbolic structures, or the basin of attraction for any attractors. These are challenging mathematical problems where any conclusions of this type lead to new insights about the development of social contracts and patterns.

[5]As already mentioned, this is not hypothetical; the work of J. Henrich [14] demonstrates this behavior in some societies.

There are many other settings beyond this particular issue of the development of social patterns and contracts. Many come from mathematical biology; e.g., see May [21] and Hofbauer and Sigmund [15] and the many references they cite. Still other concerns involve social evolution; here there are books by Frank [10] and Skyrms [62] while other approaches come from economics; here I suggest the books by Samuelson [56], Weibull [74], and Young [76] among many others. The point is, this mathematical development is in an early stage of growth.

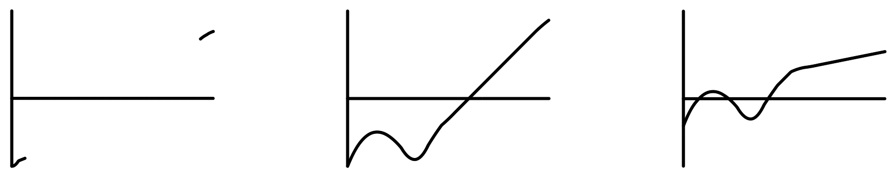
With the expansion of this area of evolutionary game theory, other concerns have emerged. For instance, is the replicator dynamics appropriate for all situations, or do other settings demand other kinds of dynamics? As an illustration, while the Ultimatum game evokes a 50-50 split in North American and Europe, there are societies where a common split is, say, around 75-25. By going to some of these cultures (more specifically, in the Amazon in Peru) to carry out Ultimatum game experiments, J. Henrich [14] discovered this significant difference. Yet it is important point to note that these different cultures experience a similar phenomenon of a threshold point, but with differing locations of equilibrium point. From a mathematical perspective, this just means that the dynamics of Eqs. 4.3, 4.4 form just needs to be replaced with a dynamical setting more appropriate for this culture. The point, however, is that different dynamics are needed.

4.4. **More general dynamics through winding numbers.** An important question is to understand the appropriate choice of dynamics. While the commonly used replicator dynamics offers a story about how different species prosper, other choices of dynamics seem to be more appropriate for other settings. For instance, consider the "best response" dynamics where each agent sizes up the current situation and then "optimize" by choosing their "best response." Or, maybe a cultural change is accomplished by an opportunistic attitude where players look around so they can mimic the strategies used by the more successful players. Indeed, the number of reasonable, different kinds of dynamics is without limit. But, is this the source of an infinite number of Ph. D. theses, or can we find a basic description of what happens to wide classes of these behaviors?

A closely related issue involves the structural stability of these results. While I do not carefully explore this issue here, the commentary of Sect. 3.7 underscores the importance of finding results which persist with perturbations of the model; this is called structural stability. (See, for instance, Robinson [35].) What can we say about the structural stability of these models? These and several other issues can be addressed by using "winding numbers." A brief outline is given here; the reader is referred to Saari [49] for a more complete description.

4.4.1. *One dimension and the average slope.* While many of the dynamical stories offered for evolutionary game theory are reasonable on a qualitative level, it is difficult to accept the precise equations. In Fig. 8, for instance, it is reasonable to accept that if one product has a near monopoly on the product, its market dominance will only grow. Similarly for the ultimatum game, it is easy to accept that if most of the population demands either 2/3 or 1/3 of the total available resources, the other species will prosper by having more to eat; but the precise dynamics are difficult to stomach. The question is to determine whether this local, readily acceptable information suffices to provide global information.

To show that the power of this local information, assume that the dynamic is continuous. Representing "moving to the left" as negative and "moving to the right" as positive, when the market story is graphed, we start with the minimal information provided by Fig. 10a where all we know is that the graph starts at a negative value on the left (reflecting the movement to the left) and ends at a positive value on the right (reflecting the movement to the right). It is trivial to see that no matter how the graph is completed to connect these endpoints, it *must* cross the x-axis.



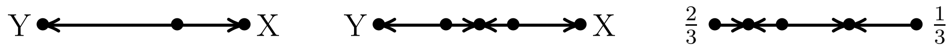
a. Local information **b.** One choice **c.** Another choice

Fig. 10. Slope conditions

Even more; in order to connect the endpoints, in general the slope of the connecting curve must be positive; it must have the sign $+1$. In turn, generically the *sum of the signs of the slopes of the curve* when passing through the x-axis must equal this global slope sign of $+1$. The reason is clear; if after the curve crosses the x-axis (which means that the slope is positive so it has the sign $+1$) it eventually moves downwards as in Fig. 10c to cross the x-axis again (where the slope must be negative or -1), then to meet the right-hand endpoint, the curve has to eventually cross the x-axis at least one more time (with a positive slope, or $+1$). In other words, each time the curve crosses the x axis downwards (with the sign of the slope -1) it has to go back upwards to cross the axis again with a positive sign of the slope $+1$. All of these signs from pairs of crossings cancel leaving the sum of the signs equaling the global value of $+1$.

The graph of Fig. 10b is purposely designed to raise a concern; what if this graph is raised so that the left-hand bump just touches the x-axis? In this case, there are only two equilibria where the first one has the sign 0. This equilibrium setting underscores the word "generic." Here, a slight change in the positioning of the curve either moves the graph upwards — causing a situation such as in Fig. 10c where the degenerate equilibrium splits to become two equilibria, or it disappears as in Fig. 10a.

At this stage, we arrive at the message of this section. Stated in words, just by knowing the local information and that the change is continuous, while we do not know how many equilibria there might be, we do know that there is at least one, and, generically, there are an odd number of them. Moreover, because the sign of the curve at each crossing of the x-axis must alternative, we know that, generically, the stability characteristic of the equilibria must alternate between being stable and unstable. All of this is indicated in Fig. 11. Namely, surprisingly minor amounts of local information provides considerable qualitative information about the global dynamics.



a. Single threshold **b.** Pocket **c.** Ultimatum effects

Fig. 11. Dynamics

To illustrate how this mathematics provides new insight, it now tells us that the message of market competition described earlier and illustrated with Fig. 8a is *not* dependent upon the specific dynamics of the earlier models. Instead, if we accept the notion of continuous change and that whatever the market dynamics may be, whenever one product exerts essentially a complete monopoly on the market it will drive the other product out of the market, then we immediately know there will be a threshold effect. The difference is, as illustrated by comparing Figs. 8a and 11a, we do not know where the threshold point is located. But, the existence of the threshold for any kind of dynamic is more important information; it tells us, for instance, that to break the monopoly, the other product must gain sufficient market share — by giving computers to schools, promotions — to cross the threshold. Moreover, we must expect the exact location of the threshold to depend upon the market characteristics of the product; cookies, computers, and cars provide different markets.

We learn even more; from Fig. 11b, we discover that this local information can be accompanied by more interesting kinds of market dynamics. Whatever form the dynamics assumes, we now know its basic characteristics; it has to introduce new pairs of equilibria where one is an attractor and the other is a repellor. The simple case offered in Fig. 11b introduce two more equilibria. This means, somewhat like what we actually see with the Apple and PC competition, that the market pressures could create a pocket reaching an equilibrium status between the two products. To break out of this pocket and achieve dominance of the market, there are two other threshold points. Thus, something other than market pressures are required to move the dynamic over the threshold.

For exposition purposes I introduced the approach with an upward sloping line which captures the sense that at both endpoints the local dynamic is moving toward the endpoint. There are three other possibilities:

1. At both endpoints, the dynamic is moving toward the left.
2. At both endpoints the dynamic is moving toward the right.
3. As in the Ultimatum game, at each endpoint the dynamic is moving away from the endpoint.

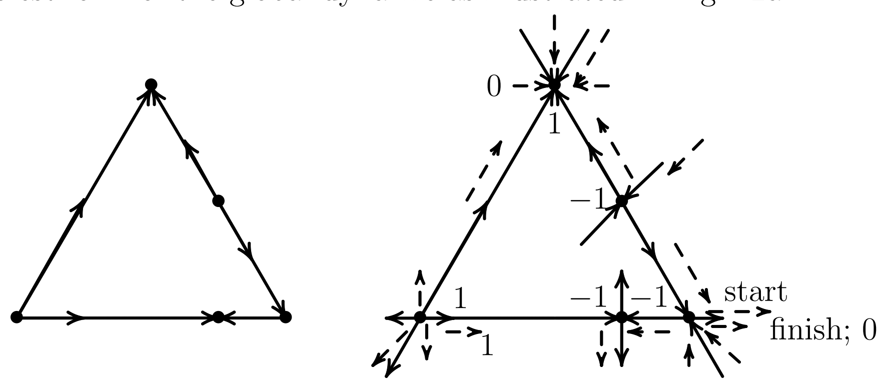
Using the same analysis we find that, generically, for the first two settings, either the complete dynamic keeps moving in the same direction, or pairs of equilibria are introduced where one is an attractor and the other is a repellor. For the last setting, the simplest picture resembles Fig. 8b except that the location of the equilibrium can differ from the middle; it's actual location depends upon the dynamic. As we know from experimental evidence, this is the case. Again, more interesting dynamics lead to more complicated settings. For instance, the next level of complexity leads to Figure 11c gives a more generalized story of the Ultimatum game.

4.4.2. *Higher dimensions and winding numbers.* We would have a powerful tool if the same kind of analysis would extend to higher dimensions. It does. For present purposes, however, I only describe what happens for two-dimensions where an easily used and intuitive tool can be introduced.

To start, suppose we have a biological, or an economic, or a social setting involving three kinds of agents. Suppose it makes sense to talk about any pair of these kinds of agents without the third. The description of the dynamic, then, can be captured on an equilateral triangle representing the simplex

$$\{(x, y, z)\,|x + y + z = 1;\; x, y, z \geq 0\}.$$

Again, local dynamics near the endpoints provides a picture of the global interactions of the pairs. This is indicated in Fig. 12a where I include all of the dynamics offered by the above; i.e., the bottom axis has something resembling the Ultimatum game, the right edge has something resembling the Market Pressure game, and the left edge is where both endpoints have the dynamic moving in the same direction. For simplicity, I use the simplest form of the global dynamic as illustrated in Fig. 12a.



a. Local information **b.** Computing

Fig. 12. Winding numbers

Even using the simplest dynamic on the edges, something has to be done about the two new equilibria. We need a local analysis to indicate what happens should a mutant, or a small number of the third kind of agent, be present. In Fig. 9, for instance, the replicator dynamics is such that near the equilibrium in the middle of the bottom edge, a small number of people who demand half of what is available would go into extinction. Presumably, this is because there is a reasonably high likelihood they will interact with people who demand two-thirds of everything. Consider, however, what might happen if this equilibrium had been more to the right with a population where a larger fraction of people demand only a third of everything. In this setting, the mutant wanting half could be expected to interact more often with those wanting only a third so they would flourish. This would be denoted by an arrow pointing upwards rather than downwards.

Namely, at each of the new equilibria an analysis based on the economics, game theory analysis, policy changes, etc., must be made to determine what happens if a small number of the new type are present; do they flourish or do they suffer? In Fig. 12b, I selected a setting where they flourish for the new equilibrium on the bottom edge and another type becomes extinct with the new equilibrium on the right edge. This describes all of the solid arrows inside the triangle and on the edges. For purposes of the next step, the arrows at each of these equilibria are symmetrically extended outside of the triangle. Namely, if a solid arrow points toward an equilibrium from the simplex, place a symmetric and companion solid arrow from outside the simplex pointing toward the same equilibrium; if the original arrow points away, have the companion arrow pointing away. All of this is indicated *with the solid arrows* in Fig. 12b. For the moment, ignore the dashed arrows.

All local information now is provided. The next step is to characterize all possible choices of the global dynamic inside the simplex. To accomplish this goal, we need something which replaces the slope in the Fig. 10 analysis. This "slope" analysis fails if only because such a representation requires replacing the line with a two-dimensional surface and the two-dimensional setting for the graph with a four-dimensional one. Fortunately, something called the "winding number" (see Milnor [28] for a description) captures what we need; this is where the dashed arrows come into the story.

Everything takes place along a curve outside of the triangle. When traveling along this curve in a counterclockwise direction, compute the number of times the "arrows" make a complete revolution in a counter-clockwise direction. Since we start and end at the same place, this number of revolutions must be an integer. Of course, if the rotations are in a clockwise direction, the answer will be negative.

Perhaps the quickest way to introduce what happens is to use Fig. 12b to compute the winding number of this example. Because the figure already is complicated enough, I purposely did not include a path, so imagine one which is very close to the outside of the triangle; say a sixteenth of an inch away. At the indicated starting point, the dynamic is moving outwards to the left; this horizontal dashed arrow is the base direction for comparisons. A way to compute this number is to put a pencil on this dashed line with the point in the direction of the arrowhead. Then, move the pencil around the triangle — actually, a little ways off of the edge — with the pencil's point always pointing in the direction indicated by the arrows. The goal is to count the number of times the pencil rotates during this journey.

The dashed arrows indicate the directions the pencil is pointing at different locations. While the pencil moves around, notice that when it is slightly to the left of the top vertex, it is pointing in the same original direction. But, to get there, the pencil did not make a complete revolution, so the computation of the winding number up to this point is zero as indicated by the dashed arrow. The next location with a horizontal arrow facing the right is slightly to the right of the lower left-hand vertex. To get to this location, the arrow spun outwards a full revolution in a counterclockwise manner; this is indicated by the count of 1 placed by the dashed arrow.

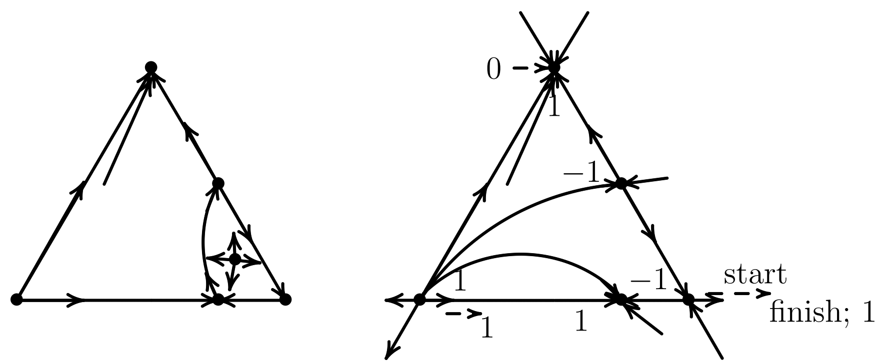
The next location where the dashed arrow is horizontal and pointing to the right is at the finish line. Notice, the pencil now winds to create a complete revolution in the opposite clockwise direction. Consequently, it has unwound the earlier "+1" revolution to end up with zero. The final winding number of zero is specified in the figure.

To use the winding number we now need to compute an index for each equilibrium. This index will be the product of the signs of the two arrows at each equilibrium; -1 indicates moving toward the equilibrium while $+1$ indicates moving away. For instance, the equilibrium **e** at the top vertex has the arrows moving toward it, so the index is $\iota(\mathbf{e}) = (-1)(-1) = 1$. Similarly, the index for the equilibrium on interior of the bottom edge is $\iota(\mathbf{e}) = (1)(-1) = -1$. The index for each of the five Fig. 12b equilibria are specified.

Similar to where the sum of the sign of the slopes at each equilibrium has to equal the sign of the global slope, the amazing relationship between the winding number and the sum of the indices is that, generically,

$$\text{Winding number} = \sum_{equilibria} \iota(\mathbf{e}). \tag{4.5}$$

4.4.3. *Global dynamics.* To illustrate Eq. 4.5, recall that the winding number in Fig. 12b is zero. Hence, the sum of all indices for all equilibria must cancel to equal zero. But, this is not the case as the sum equals -1. This means we are missing at least one equilibrium with index $+1$. But to have an index of $+1$, either we are discussing an attractor or a repellor.



a. "One" general story **b.** Altered story

Fig. 13. Global dynamics

Figure 13a describes one of the only two possibilities. Notice how most of the trajectories end up at the top vertex. In other words, the huge region to the left of the curved line is the "basin of attraction" for the top vertex; all solutions in this region must head toward the top vertex where two of the three kinds of agents are eliminated. Some of the dynamics, such as those starting near the bottom edge and near the left-hand vertex illustrate an interesting and complicated behavior before the bottom two kinds of agents are eliminated. This is because the behavior first must move near the bottom edge — suggesting the near demise of the third type — until it approaches the interior equilibrium on this edge. At this point, the trajectory slides up near the upward bound arc until it approaches the equilibrium located on the interior of the right-hand edge. At this point the motion changes behavior again to move upward toward the top vertex.

The interesting behavior is in the region near the right-hand vertex. Inside of this region is the missing equilibrium. Remember, there are two choices for this equilibrium; a repellor or an attractor. The selection made for Fig. 13a is a repellor. Notice what must happen; the motion has to move out from the new equilibrium and approach the boundaries of this region. The boundaries, however, force the motion to cycle in the indicated manner. Consequently all non-equilibrium motion in this small region is forced toward the boundary where it cycles in a clockwise manner — forever. In terms of the three kinds of agents, we encounter periods of time where each of two types appears to approach extinction in flames only to adopt the phoenix resurrection and exist again for another period of time.

The other possible story is where this new equilibrium is an attractor. Here, instead of all motion in this corner pushing out from the boundary, all motion moves inwards toward an equilibrium forcing a position of coexistence of the three types. There are, of course, more complicated dynamics associated with this local story. Namely, just as in Fig. 10c, it is possible for another pair of equilibria to emerge. If this happens, one must have index $+1$ while the other has index -1. The question is to understand where to locate these new equilibria. Both have to be either inside the large, or the small, region. The explanation for this can be derived from the argument given next.

4.4.4. *Restriction on dynamics.* I now must support my comment that these two stories are the only ways to introduce a single equilibrium of index $+1$. The argument involves how the stable and unstable manifolds connect. Each manifold is a line; lines have to go on forever, as with the spiraling lines in the small region of Fig. 13a, they had to connect creating a circle, or they must have an endpoint which is another equilibrium. For reasons I will not explain here, only the first and last are viable.

Armed with this information, suppose we have an attractor. (With minor changes, the same analysis holds for a repellor.) This attractor allows the unstable manifold from the bottom edge to terminate in the attractor, but it creates a problem about the positioning of the stable manifold from the right edge — where does it terminate (going backwards)? It cannot come from the attractor; it cannot come from the equilibrium which is the lower right hand corner, so it must come from the left hand vertex repellor. If so, then much as with the line in Fig. 13b, it would create a line starting in the left-hand vertex and terminating in the midpoint of the right edge. To find what can go wrong, compute the winding number along a different arc; start near the right hand vertex, go to the midpoint of the right edge, move through the triangle along this new line which connects to the left hand vertex, and then return to the starting point along the bottom edge. By doing so, the winding number is $+1$. But, the sum of the existing equilibria on the edges (since we do not count the top vertex as it is "outside" our curve) is -1; consequently, this setting requires adding not one, but two equilibria in the new region with the sum of their indices equaling $+2$.

We now know that the unstable manifold from the bottom edge is the stable manifold of the equilibrium on the right edge. But where do we locate the new equilibrium of index $+1$? For the two regions, compute the winding number of each. When this is done, we discover that the new equilibria must be in the small region on the right. Namely, the winding number is a powerful tool that can be used in an iterative fashion to identify where equilibria must be located.

4.4.5. *Change in the analysis.* As another example, consider only the local information from each of the three vertices of Fig. 9 — this is the Ultimatum game analysis. Using the analysis about lines as captured by Fig. 10, the simplest story along each edge is that there are no equilibria along the side edges, and there is one equilibrium on the bottom edge. If this bottom edge equilibria is located near the middle, or to the left, it means that the kinds of people demanding a half are small in population and have a strong likelihood of being matched with an individual demanding two-thirds. This means that the ones requiring a half have a strong likelihood of being driven out of existence; i.e., a "local" analysis suggests that the motion must move inwards as displayed near the bottom edge of Fig. 9.

This local information leads to a winding number of +3 while the sum of the indices over the four equilibria on the edges is +4. Thus at least one extra equilibrium with index −1 must exist in the interior; namely, this equilibrium has a stable and an unstable manifold. By using the above argument about stable and unstable manifolds — that they are lines which must terminate in another equilibrium or expand forever — the only possible picture is essentially that of Fig. 9. Qualitatively, it is the only picture for adding a single equilibrium, but the location of the equilibrium could be anyway within the interior of the triangle. Different locations, of course, change quantitative conclusions about the size of the basin of attraction; e.g., do "most" trajectories tend to the middle equilibrium on the bottom edge (which happens if the new equilibrium is closer to the top vertex), or do most trajectories tend to the top equilibrium (which occurs if the new equilibrium is closer to the bottom edge)?

Remember, since Fig. 9 is based on a particular equation, it need not tell us everything that could occur. To explain, suppose we have a society whereby the equilibrium on the bottom edge is nearer the right-hand vertex. This location means that a mutant demanding half of the pie is more likely to encounter agents who want only a third of it. This more favorable location suggests that the "halfers" will prosper, rather than dying out. In other words, the equilibrium on the bottom edge of Fig. 9 now has index −1. In turn, the changed nature of this equilibrium reduces the winding number to +2 — a value which agrees with the sum of the indices along the edges. Here the simplest model associated with this local information *has no interior equilibria!* Instead, all trajectories terminate at the top vertex; in this story, where the location of the bottom trajectory does not lead to an annihilation of those wanting a half, the basin of attraction for the one-half type is everything in the interior of the triangle.

4.4.6. *Policy.* A powerful use of this "winding number" tool is that it allows us to explore the potential impact of "policies" — economic policies, social policies, political policies — without having to formally define the policy. Instead, all we need is to understand the local effects of a policy at certain crucial junctures rather than all of its ramifications.

To suggest what I mean, consider the about analysis about what would happen if the local behavior of the middle bottom equilibrium of Fig. 9 would change. A fairly dramatic consequence is that the simplest story would require all behavior to gravitate toward the desired situation where everyone demands their share of a half. So, if instead of an argument involving replicator dynamics, the change occurred because of a policy decision, we would expect one global consequence to be a global change in attitudes. (More complicated consequences would involve if pairs of equilibria would be introduced.)

To further describe this effect, return to Fig. 13a where the dynamic along the bottom edge is motivated by standard competition behavior as manifested by competition between PC and Mac. Suppose the figure displays what happens with a third kind of computer. Policy changes may allow the behavior near the bottom equilibrium to provide added advantages for the existing two computer types. The question is to understand the more global consequences of this policy.

Again, the analysis is to compute the winding number. As illustrated with the dashed arrows placed on the outside of the triangle in Fig. 13b, the winding number now is +1 — a value which coincides with the sum of the indices of the equilibria along the edge. Consequently, the simplest picture of global dynamics does not require an interior equilibria! By using the arguments above about how to connect the stable and unstable manifolds, the global dynamics must assume a qualitative picture of the kind given by Fig. 13b.

It is interesting how this one small local change has significant global consequences. For instance, with the original local analysis of the policy near the bottom interior equilibrium, the motion either spiralled around the right-hand region, or there was an interior equilibrium where all three brands shared different amounts of market share. The policy change, however, creates two major equilibria; both of them require at least one of the brands to become extinct. The line connecting the left-hand vertex to the vertex of the middle of the right-hand line creates two basins of attraction. All starting positions below this line must gravitate toward the interior bottom equilibrium where one type is driven out of business; all starting points above this line must tend toward the top vertex where only one brand survives.

4.5. **Conclusion.** As asserted, evolutionary game theory is a new topic with significant consequences. My emphasis is on my introduction of winding numbers to understand the qualitative aspects of the analysis, and even this is just a hint of what can happen. (A fuller description will appear elsewhere.) But, there are many other topics, many other approaches from mathematics which will significantly enrich this area. It most surely is a growth area!

5. Adam Smith's "Invisible hand" — and continuous foliations[6]

For the last section of these notes, let me turn to something very different; the behavior of prices. This is, of course, one of the a basic issue of economics. My approach is via dynamics.

One of the lessons learned from modern dynamics is that natural systems can be surprisingly complex. No longer are we astonished to discover that systems from biology (e.g., May [20, 21], Hofbauer and Sigmund [15]), or the Newtonian N-body problem (e.g., Mather and McGehee [22], Moser [29], Xia [77], Saari and Xia [55]) give rise to highly complicated, chaotic behavior. Yet, this seeming randomness sharply contrasts with what we have been conditioned to expect from economics. After all, on the evening news, in newspapers, and calls from politicians during economic difficulties condition us to expect that if we just leave market forces alone, market pressures will drive prices

[6] This section heavily depends on the paper Saari[39].

toward an equilibrium experiencing the desired status of balance between demand and supply.

What a nice story. But is it true? I do not know, nor does anyone else. In part, this is because no general economic theory, no mathematical theory exists to justify it. Quite to the contrary, what we do know indicates that even the simple models from introductory courses in economics can exhibit dynamical behavior far more complex than anything found in classical physics or biology! In this concluding section, I explain why this is the case.

Before turning to the story, let offer a mathematical challenge. The mathematical description given below, which is needed to analyze the Adam Smith story, has disturbing consequences about economics; it shows that all sorts of wild economic behavior can occur. Do I believe this? In part, yes; for instance, the mathematical arguments are correct (I have to say this because many of them are mine). But I am not confident that the various results represent economic reality. So, the mathematical challenge is to understand why basic economic assumptions lead to such disturbing economic conclusions; conclusions which may be false for economics. (Some preliminary work in this direction can be found in Saari [45, 48].)

5.1. **The traditional story from economics.** Consider a specialized setting consisting of $c \geq 2$ commodities where the agents can exchange goods according to (positive) prices but there is no production; this is called a *pure exchange economy.* If $p_j > 0$ is the price per unit of the jth commodity, then the cost of $x_j > 0$ units is $p_j x_j$. This means that if $\mathbf{p} = (p_1, \dots, p_c)$ is the price vector over all commodities, then the cost of a *commodity bundle* $\mathbf{x} \in R^c_+$ is given by the scalar product $\mathbf{p} \cdot \mathbf{x}$.

Since this is a pure exchange economy, the only way agents can get money is to sell what they start with; this is called the *initial endowment* $\boldsymbol{\omega}_j \in R^c_+$. Thus, at prices $\mathbf{p}$, the jth agent has the wealth $\mathbf{p} \cdot \boldsymbol{\omega}_j$. With this wealth, the agent can afford to buy a commodity bundle $\mathbf{x}_j \in R^c_+$ which satisfies the budget constraint

$$\mathbf{p} \cdot \mathbf{x}_j \leq \mathbf{p} \cdot \boldsymbol{\omega}_j. \tag{5.1}$$

Presumably, all of the goods are desirable, so equality in Eq. 5.1 will be achieved. This equality setting, which is given by the plane

$$\mathbf{p} \cdot (\mathbf{x} - \boldsymbol{\omega}_j) = 0 \tag{5.2}$$

which passes through $\boldsymbol{\omega}_j$ with the price vector $\mathbf{p}$ as a normal, describes what is called the *budget plane*. This is depicted by the budget line in Fig. 14. At the given prices $\mathbf{p}$, the jth agent wishes to maximize his needs or pleasure by selecting a personally optimal choice along the budget plane. The next part of the story describes how this is done.

Each person optimizes according to personal preferences. Because a natural ordering does not exist on R^c for $c \geq 2$, an ordering is imposed by assuming that each person's preferences are given by a *utility function* $u_j : R^c_+ \to R$ where $u_j(\mathbf{y}) > u_j(\mathbf{x})$ if and only if the jth agent strictly prefers commodity bundle $\mathbf{y}$ to $\mathbf{x}$. Two of the level sets for a particular choice of u_j are represented in Fig. 14.

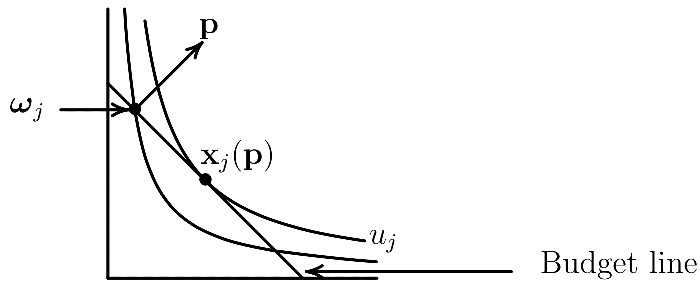


Fig. 14. Finding jth agent's individual demand

To simplify the analysis, it is traditionally assumed that all components of ∇u_j are positive and that u_j is strictly convex. The first assumption means that "more is preferred to less." The second assumption is imposed to ensure unique solutions when an agent optimizes. To describe this condition geometrically, for any $\mathbf{x}$ consider the hyperplane passing through $\mathbf{x}$ with $\nabla u_j(\mathbf{x})$ as a normal. This strict convexity condition requires the level set of u_j passing through $\mathbf{x}$ to meet the hyperplane only at $\mathbf{x}$; the rest of the level set is on the same side of the hyperplane as the gradient $\nabla u_j(\mathbf{x})$.

The optimal choice for the jth agent given the prices and the initial endowment now should be clear; as depicted in Fig. 14, it is the unique $\mathbf{x}_j$ where a level set is tangent to the budget plane. Applying Lagrange multiplier techniques to this idealized set-up, the jth agent's optimal choice is the commodity bundle $\mathbf{x}_j$ on the budget plane where

$$\mathbf{p} = \lambda_j \nabla u_j(\mathbf{x}_j) \tag{5.3}$$

for some positive scalar λ_j. The strict convexity assumption requires that the $\mathbf{x}_j$ choice is unique for each $\mathbf{p}$; represent this fact with $\mathbf{x}_j(\mathbf{p})$. The jth agent's excess demand is the difference between what the agent wants, $\mathbf{x}_j(\mathbf{p})$, and what the agent will supply, $\boldsymbol{\omega}_j$. It is

$$\xi_j(\mathbf{p}) = \mathbf{x}_j(\mathbf{p}) - \boldsymbol{\omega}_j. \tag{5.4}$$

This elementary derivation of $\xi_j(\mathbf{p})$ leads immediately to the basic properties of the *aggregate excess demand function* — the sum of all individual excess demands —

$$\xi(\mathbf{p}) = \sum_{j=1}^{a} \xi_j(\mathbf{p}).$$

These three basic properties are known as *Walras' laws.*

1. $\xi(\mathbf{p})$ is single valued and smooth. (This follows from the strict convexity assumption on preferences and smoothness of the utility functions u_j.)
2. $\xi(\mathbf{p})$ is homogeneous of degree zero. (In the derivation of the $\xi_j(\mathbf{p})$ for each agent, notice that the only role played the price vector $\mathbf{p}$ is to define the budget plane. This plane, and the choice of ξ, remains the same for any scalar multiple of $\mathbf{p}$. Thus, for any $\lambda > 0$, $\xi(\lambda\mathbf{p}) = \xi(\mathbf{p})$.)
3. $\xi(\mathbf{p})$ is orthogonal to $\mathbf{p}$. (Since, by construction, $\xi_j(\mathbf{p}) \cdot \mathbf{p} = 0$ for each j, the same orthogonality relationship holds for the sum $\xi(\mathbf{p}) = \sum_{j=1}^{a} \xi_j(\mathbf{p})$.)

5.2. **Flow on a sphere and the existence of equilibria.** In order to describe this setting in mathematical terms, notice how condition 2, the homogeneity of $\xi(\mathbf{p})$, allows the price vector to be scaled. A traditional choice adopted by economists is to set one of the prices equal to unity. Another choice, which is used here, is to require the prices to satisfy

$$||\mathbf{p}||_2 = \sqrt{\sum_{j=1}^{c} p_j^2} = 1.$$

In other words, the price simplex becomes the portion of the unit ball which is in the positive orthant of R^c_+; this is denoted by S^{c-1}_+. For simplicity, rather than drawing the surface of the ball, only the "flattened" version of this simplex — given by the equilateral triangle of Fig. 15 — is used.

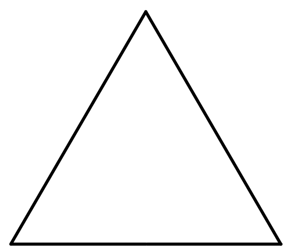


Fig. 15. The price simplex

The third listed Walras' property — the orthogonality of $\xi(\mathbf{p})$ and $\mathbf{p}$ — means that $\xi(\mathbf{p})$ defines a tangent vector field to this S^{c-1}_+ portion of the sphere. Consequently, economic issues about how prices are formed and change translate into mathematical questions about the properties of flows generated by tangent vector fields on a sphere. The natural mathematical question is to ask, *which* tangent vector fields? I will return this question in the next subsection.

I have yet to indicate a use of the remaining Walras' property that $\xi(\mathbf{p})$ is smooth. To do so, recall that the basic concern of this section is to understand whether Adam Smith's invisible hand story has any validity. Important contributions were made by K. Arrow and G. Debreu [4]. (Also see Arrow and Hahn [3] and Debreu [8].) To make the problem more tractable, Arrow and Debreu separated the dynamics of the "Invisible Hand" from the question about the existence of a price equilibrium. That is, they asked whether there exists a $\mathbf{p}^*$ where $\xi(\mathbf{p}^*) = \mathbf{0}$.

To understand why such a $\mathbf{p}^*$ must exist, go back to Fig. 14 and consider a budget line which is nearly horizontal; i.e., one of the prices is nearly zero. If follows from the assumptions placed on preferences — in terms of the utility functions — that a agent's optimal choice must be far to the right. Stated in economic terms, this means that if the price for a desired commodity is nearly zero, the commodity will attract a huge demand. Restating this condition in terms of the aggregate excess demand function, it means that $\xi(\mathbf{p})$ will point toward the interior of the price simplex all along the boundary of the simplex. This situation is indicated in Fig. 16.

According to the Brouwer fixed point theorem (e.g., see Milnor [28]), the combination of the continuity of $\xi(\mathbf{p})$ with its boundary behavior on the simplex means that there exists a point $\mathbf{p}^*$ where $\xi(\mathbf{p}^*) = \mathbf{0}$. (As an alternative argument, compute the winding number; it requires at least one equilibrium with index +1.) Thus, price equilibria do exist. This simple argument captures the essence of the argument developed by Arrow and Debreu which holds for more general economic settings.

5.3. **What else can happen?** To see how the above leads to natural mathematical questions, let $\Xi(S_+^{c-1})$ represent the space of continuous tangent vector fields on S_+^{c-1}, and let $\mathcal{U}^c$ be the space of continuous, strictly convex functions

$$u : R_+^c \to R.$$

In mathematical terms, the above traditional story defines a mapping

$$\mathcal{F} : [\mathcal{U}^c \times R_+^c]^a \to \Xi(S_+^{c-1}) \tag{5.5}$$

where the exponent a represents the product over all $a \geq 2$ agents and the R_+^c term captures the position of the initial endowment. Again, just by representing the economic description in the Eq. 5.5 form generates several natural mathematical questions. The most obvious one is to characterize the image of $\mathcal{F}$.

5.3.1. *Economic and mathematical representations.* In economic terms, this question about the image of $\mathcal{F}$ was first raised by the economist Hugo Sonnenshein [67, 68]. In his terms, he wondered whether the aggregate excess demand function $\xi(\mathbf{p})$ had any general properties other than those captured by the above Walras' laws. While this was not an issue raised by Sonnenshein, to motivate his question, suppose that the Adam Smith story always is true. This would mean that the market pressures always pull the prices toward a price equilibrium. If so, then another general property enjoyed by all choices of $\xi(\mathbf{p})$ would be that it has a global attractor; there is a point $\mathbf{p}^*$ toward which all solutions from either the continuous dynamic

$$\mathbf{p}' = \xi(\mathbf{p}), \tag{5.6}$$

or from the discrete, iterative dynamics

$$\mathbf{p}_{n+1} - \mathbf{p}_n = \xi(\mathbf{p}_n) \tag{5.7}$$

would converge.

The surprising and beautiful result, which Sonnenshein [67, 68] initially proved in a restricted setting, R. Mantel [19] extended to a more general setting involving smooth utility functions for a sufficient number of agents, and G. Debreu [9] proved with the sharp conclusion requiring only $a \geq c$ agents with strictly convex, continuous preferences, is that $\xi(\mathbf{p})$ *has no other general properties.* To be more precise, the Sonnenshein, Mantel, Debreu Theorem (the SMD Theorem) bounds all prices away from zero. So, select $\epsilon > 0$, and let

$$S_\epsilon^{c-1} = \{\mathbf{p} = (p_1, \dots, p_c) \,|\, ||\mathbf{p}||_2 = 1,\ p_j \geq \epsilon \text{ for all } j = 1, \dots, c\}.$$

For $c = 3$, this set is represented in Fig. 16; it is the set of all prices inside the ϵ boundary — the slender edge of the triangle. With this slight, technical constraint, the SMD theorem is as follows.

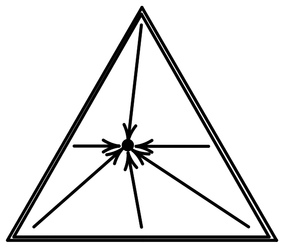


Fig. 16. The price simplex S^2_ϵ

Theorem 5. *(SMD Theorem.) For $c \geq 2$ commodities, let $f(\mathbf{p})$ be any continuous function satisfying Walras' laws; i.e., $f(\mathbf{p})$ is continuous, $f(\mathbf{p})$ is homogeneous of degree zero, and $f(\mathbf{p})$ is orthogonal to $\mathbf{p}$. If there are $a \geq c$ agents, a pure exchange economy exists so that on S^{c-1}_ϵ, $\xi(\mathbf{p}) = f(\mathbf{p})$. However, for $a < c$ agents, there exist choices of $f(\mathbf{p})$ where such an economy cannot be constructed.*

Re-expressing this theorem in the mathematical terms which are suggested by the Eq. 5.5 representation, the Sonnenshein-Mantel-Debreu theorem becomes

Theorem 6. *(SMD Theorem.) For $a \geq c$ agent and $\epsilon > 0$, the mapping*

$$\mathcal{F} : [\mathcal{U}^c \times R^c_+]^a \to \Xi(S^{c-1}_\epsilon) \tag{5.8}$$

is surjective. If $a < c$, then the mapping of Eq. 5.8 is not surjective.

Just think what this theorem means! As pointed out by Saari [39] (much of this section is heavily based on this reference), you can visit your local mathematics or physics department to find the newest, most complicated version of chaos involving either an continuous differential equation

$$\mathbf{x}' = f(\mathbf{x})$$

or an iterative equation

$$\mathbf{x}_{n+1} = \mathbf{x}_n + f(\mathbf{x}_n)$$

and you are assured that a "simple" pure exchange economy exists where either the price dynamics of Eq. 5.6 or 5.7 form exhibits the precise, same chaotic effect. Simple economics can be surprisingly complicated. Indeed, as shown in Saari [45, 46], the preferences assigned to economic agents which result in chaotic behavior can be quite innocuous and of the kind found in many papers in economic journals.

5.3.2. *Other questions based on the mathematics.* Once a mathematician sees the SMD Theorem expressed in the Thm. 6 formulation, several mathematical questions jump immediately to mind. The following provides just three obvious samples.

1. If the image of $\mathcal{F}$ is not surjective for $a < c$, then what is the image?
2. We now know that the image of $\mathcal{F}$ is surjective for $a \geq c$. OK, let us push the situation. For instance, following the lead of the results in Section 2, what happens to the image when the mapping $\mathcal{F}$ is extended? What does it take before the image of extensions of $\mathcal{F}$ are not surjective?

3. Is there a simple mathematical plausibility explanation which provides intuition for this SMD theorem?

Although question # 1 clearly is interesting and important, I do not believe it has been answered, but, as suggested later, partial results can be obtained. Indeed, providing a partial answer to # 3 offers some obvious answers for # 1. But, I suspect that a full answer for question # 1 remains to be found.

What about question # 2? The first step is to determine what constitutes a "natural extension" of the mapping $\mathcal{F}$. Here some partial answers follow. (Again, the full question remains unanswered.) The choice I describe here is to consider different subsets of the set of c commodities. For instance, suppose that legal regulations prohibit one of the commodities to be exchanged. Stories are easy to supply; this might be because the commodity is perishable and the useful date has expired, it might be software which has become dated or in a legal battle over copyright, it might be a commodity where the supplies are not available, or it could be a substance which the legal system has rendered illegal.

These examples suggest that we compare the aggregate excess demand function for the full set of c commodities with the aggregate excess demand function for all possible subsets of commodities. Based on results coming from the "revealed preference literature," one must suspect that there is some sort of relationship. As indicated by the next theorem, there is not. There need not be any relationship whatsoever between the aggregate excess demand function of the full set of commodities and any subset of these commodities.

Theorem 7. *(Saari [41].) Assume there are $c \geq 2$ commodities and let $\epsilon > 0$ be specified. For each subset of at least two commodities, let $f_{subset}(\mathbf{p}_{\subset}))$ be a function which is homogeneous of degree zero, which is at least continuous, and which is orthogonal to $\mathbf{p}_{subset}$ which uses the prices of $\mathbf{p}$ relevant for the commodities in the specific subset. For $a \geq c$ commodities, there exists a pure exchange economy where for each subset of commodities $f_{subset}(\mathbf{p}_{subset}) = \xi_{subset}(\mathbf{p})$ on $S_{\epsilon}^{(|subset|-1)}$ for all subsets of two or more commodities. If $a < c$, then there exist choices of $f_{subset}(\mathbf{p}_{\subset})$ which cannot be supported by any pure exchange economy.*

Restating this conclusion in mathematical terms, if $a \geq c$ and if $\mathcal{F}$ represents the selection of the aggregate excess demand function for all economies of two or more commodities, then

$$\mathcal{F} : [\mathcal{U}^c \times R^c_+]^a \to \Pi_{|subsets| \geq 2}\, \Xi(S_{\epsilon}^{|subset|-1}) \tag{5.9}$$

is onto. However, if $a < c$, then this mapping is not surjective.

To suggest consequences of Thm. 7, notice that it allows economic settings of three commodities, say {beer, pretzels, cheese} where the "Adam Smith invisible hand story dominates because the price dynamics requires the iterative dynamics (Eq. 5.7) of all starting points to converge to the central point $\mathbf{p}^*$ as indicated in Fig. 16. However, the iterative price dynamics of {beer, pretzels} or of {beer, cheese} fail to converge as the dynamics is "chaotic" even though the iterative price dynamics over the commodities {pretzels, cheese} is surprisingly well behaved. Once there are more than three commodities, the suggested dynamics can be surprisingly wild; they can even be supplied by cynical faculty coming from mathematics or the physical sciences.

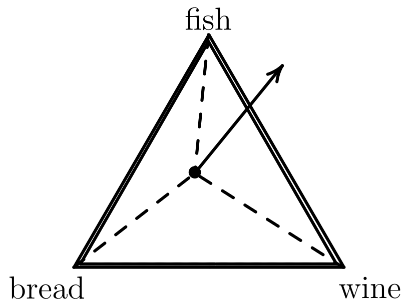


Fig. 17. Finding preferences

5.3.3. *Sense of the proof.* Let me provide some feel for why we should accept that the SMD Theorem and my extension are true; I do this by first appealing to the Adam Smith story. Recall, Smith's story allows each individual to make a specific product which can be exchanged for other products. So, suppose there are three agents; one produces bread, the other produces wine, and the last produces fish. At a particular price given by the bullet in Fig. 17, let $\xi(\mathbf{p})$ be given by the arrow. The goal is to construct preferences for the three agents which given rise to this vector.

Intuition why this can be done follows fairly directly from Smith's story. The baker, for instance, has all the bread he needs, so his wish is to have some combination of wine and fish; his particular $\xi_j(\mathbf{p})$, then, can be expected to lie in the sector defined by the shaded lines which is further from the "bread" vertex. The actual size and direction of $\xi_j(\mathbf{p})$ comes from the baker's preferences.

Similarly the fisher's and the wine-maker's choice of $\xi_j(\mathbf{p})$ are vectors in the appropriate sectors defined by the dashed lines which is furthest from their vertex. As utility functions and choices of initial endowments can be selects to make the lengths of these $\xi_j(\mathbf{p})$ vectors any desired value, it is clear that these choices can be made to support any specified vector; in particular, for this choice the baker's excess demand function dominates. Notice how this argument collapses with only two kinds of agents. For instance, if only two agents were allowed, and one is the fisher while the other is a wine-maker, then this particular choice of $\xi(\mathbf{p})$ could *not* be realized. From this explanation, a sense of what can happen; that is, a sense of what is in the image of $\mathcal{F}$, should be immediate.

Now suppose that very close to the particular price indicated by the bullet, another price $\mathbf{p}^*$ has an aggregate excess demand function $\xi(\mathbf{p}^*)$ which is essentially $-\xi(\mathbf{p})$; i.e., the excess demand function demonstrates such volatility that even a very slight price change flips the excess demand in a different direction. For this to happen, the baker's demand has to suddenly drop while that of the fisher and wine-maker suddenly jump to crave bread. How can such radical changes in behavior happen? By experimentation, it becomes clear that this can be done if the level sets are essentially straight lines — that is, of the kind often seen in journal articles.

The challenge to prove these theorems is to prove that a continuous foliation can be constructed with the appropriate properties.

5.4. **Dynamics.** As I stated, the brilliance of the Arrow and Debreu approach to this problem was to separate the question of the existence of the equilibria from the dynamics of getting there. But, as described above, we now know that the usual "supply and demand" story as captured either by the continuous story of Eq. 5.6 or the iterative story of Eq. 5.7 does not work in general.

Maybe these dynamics are overly optimistic. Maybe the actual price dynamic modifies the market pressure in a more sophisticated manner. To explore this story, suppose that the actual *modification*, represented by M, of $\xi(\mathbf{p})$ is given by $M(\xi(\mathbf{p}))$. So, with the continuous dynamic, the actual dynamic is given by

$$\mathbf{p}' = M(\xi(\mathbf{p})). \tag{5.10}$$

What is M? We don't know. It is reasonable to assume that however the market pressures behave, it is continuous and that it stops when a price equilibria is reached. this last condition is given by

$$M(\mathbf{0}) = \mathbf{0}. \tag{5.11}$$

Unfortunately, the next result, which comes from the description in Saari and Simon [53], states that no such modification affect can exist.

Theorem 8. *(Saari and Simon* [53]*.) For any continuous M satisfying Eq. 5.11 and at least as many agents as there are commodities, there is a open set of economies (i.e., an open set of utility functions and of endowments for the agents) where the price dynamics of Eq. 5.10 does not converge to a price equilibria.*

It is clear that more information about the market is needed. So, maybe the prices are governed by

$$\mathbf{p}' = M(\xi(\mathbf{p}), D\,\xi(\mathbf{p})) \tag{5.12}$$

where $D\,\xi$ is the Jacobean of ξ; it has the rate of change of each commodity with respect to price changes of each of the commodities. Clearly, something requiring all of this information is unrealistic from an economic perspective. Thus, the goal is to understand how much, or how little, of this information is needed.

To address this question, Saari and Simon divided the study of the price dynamics into two categories; one is to find how much market information is needed to converge to a specific equilibria by starting near it — we called this a *local effective price mechanism* — and how market information is needed to converge to *some* equilibria — we called this a *global effective price mechanism*. What we showed is that a local effective price mechanism required *all* of the information implicitly required by Eq. 5.12! In other words, leaving out information how one the demand for one commodity will change with respect to a price change for another commodity is enough to foreclose convergence for some economies!

Is this enough information? Actually, it is. Motivated in part by these issues of the price dynamics, Steven Smale [66] proved that it is possible to create convergent mechanisms of the Eq. 5.12 type which always converge to some equilibria. His scheme is much like a higher dimensional Newton's method.

Somewhat surprising is another result which we proved. Namely, the informational requirements for a Global Effective Price Mechanism is slightly more modest than that for a Local Effective Price Mechanism. We showed that by use of the winding number, or index theorems, mentioned in the last section, some terms in the Jacobean $D\,\xi$ could be ignored. But, not too many of them! (For the mathematicians reading this, ignoring a variable creates a line in the space of $G(n, R)$; our proof involved the geometry of this space.)

While this conclusion is negative, it is not overly negative. On the other hand, a differential equation is implicitly using an infinite amount of information coming from the time variable. Therefore, I became interested in determining whether these conclusions would hold for more realistic settings of a discrete price mechanism. This means, suppose a theory could be invented asserting that there is a method M where the price dynamics given in

$$\mathbf{p}_{n+1} = \mathbf{p}_n + M(\xi(\mathbf{p}_n), D\,\xi(\mathbf{p}_n)) \tag{5.13}$$

to converge for all standard pure exchange economies. The question is to find the minimal amount of information needed by this mechanism. The theorem (Saari [40]) asserts that no such prices mechanism exists.

Well, let's throw in more and more information until convergence does occur. We could use higher and higher derivatives; we could use information from past prices. The basic theorem states that no finite amount of information suffices.

Theorem 9. *(Saari [40]) Suppose there are at least as many agents as commodities. For any continuous choice of M satisfying Eq. 5.11 and for any integer s giving the number of derivatives and integer k giving the number of past prices, consider the price dynamic of*

$$\mathbf{p}_{n+1} = \mathbf{p}_n + M(\xi(\mathbf{p}_n), D\xi(\mathbf{p}_n), \dots, D^s\xi(\mathbf{p}_n), \dots, \xi(\mathbf{p}_{n-k}), \dots, D^s\xi(\mathbf{p}_{n-k})).$$

There exists an open set of economies and an open set of initial prices where the dynamics does not converge to any price equilibria.

Also notice that the same conclusion must hold for schemes finding zeros of analytic functions. (See Saari [42].)

I wish I could end on a more positive note. I cannot about the economics nor about Adam Smith's story. But, I can for the mathematics. After all, these results demonstrate that there is so much about the mathematics of these issues that we do not understand. The positive conclusion, then, is that there are many mathematical issues that are waiting to be explored.

6. A final comment

What you have seen is a highly personal, highly restrictive survey of some questions coming from the mathematical social sciences. My hope is that they have sufficiently intrigued some mathematicians to start examining other questions. This will become a very rich area where, as true with several areas, many of the rewards will be bestowed on some of the mathematicians who move into this area at an early stage. I welcome all of you!

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