Pacific Institute for the Mathematical Sciences

PIMS Distinguished Chair Lectures

VLADIMIR TURAEV

PIMS Distinguished Chair,
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Torsion invariants of 3-manifolds

Edited by John Bryden, Florian Deloup and Peter Zvengrowski

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Lecture 1. Introduction

Since we have a diverse audience today, including high school students from Prince Edward Island, Nova Scotia, New Brunswick, Québec, and Ontario, we will begin these lectures with some of the most elementary ideas from topology: spaces (especially manifolds) and the comparison of spaces. Let us start with some examples.

Here are some spaces that are very familiar: the real line $\mathbb{R}$, the circle S^1, the disc D^2, the 2-sphere S^2, the ball (or "3-disc") D^3. Explicitly, one can write (or take as models)

$$\begin{aligned} S^1 &= \{(x,y) : x^2+y^2=1\} \\ D^2 &= \{(x,y) : x^2+y^2 \le 1\} \\ S^2 &= \{(x,y,z) : x^2+y^2+z^2=1\} \\ D^3 &= \{(x,y,z) : x^2+y^2+z^2 \le 1\}, \end{aligned}$$

where x, y, z run over $\mathbb{R}$.

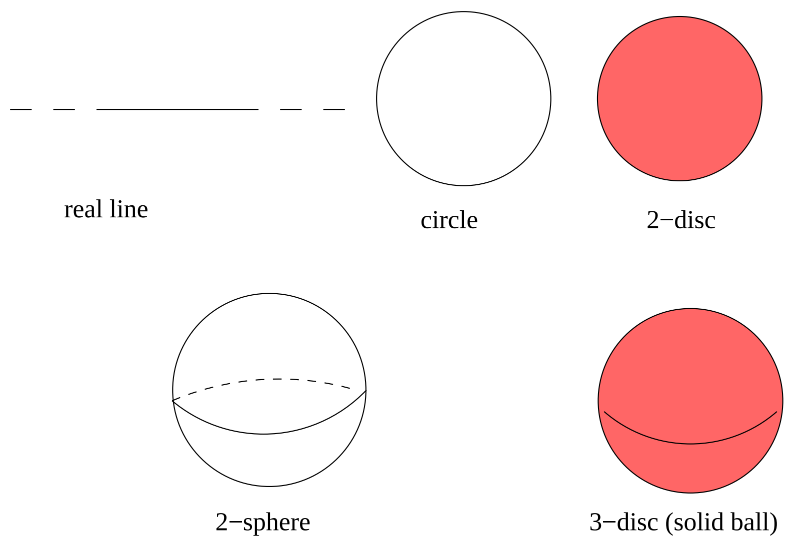


Figure 1: Some Basic Topological Spaces

Each of the above spaces has a dimension, which one can think of as the number of degrees of freedom of motion at a typical point in the space. For example, on a space like S^2, which can be thought of as the surface of the Earth, each point is specified by its latitude and longitude, i.e., there are 2 degrees of freedom. So this space is 2-dimensional. Similarly, S^1 and $\mathbb{R}$ are 1-dimensional, D^2 is 2-dimensional, and D^3 is 3-dimensional.

In topology spaces are thought of as the same if they have the same form. Thus, a circle, an ellipse, and a triangle are all equivalent topologically. Another example would be the two fingerprints shown in the following diagram. Clearly they have different form, so are different

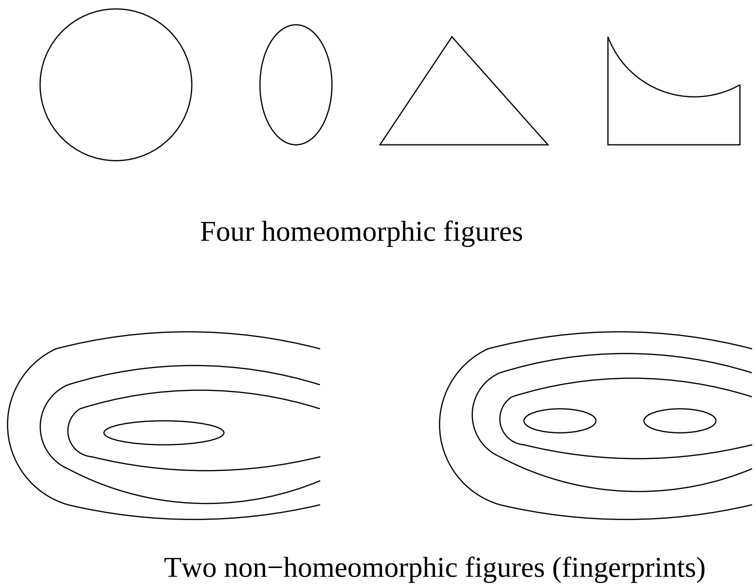


Figure 2: Homeomorphic and Non-homeomorphic Figures

topologically. Mathematicians use the word “homeomorphic” (written with the symbol $\cong$) to denote this type of equivalence, its Greek roots are:

$$\text{homeo} \ = \ \text{“similar”} = \ \overset{c}{o}\mu o\acute{\iota}os$$

$$\text{morph}{=}\text{“form”}{=}\mu\acute{o}\rho\varphi\eta.$$

Topologists look for tools to decide when two spaces are homeomorphic, such as in the next diagram, and also tools to decide when two spaces are not homeomorphic (written $\ncong$). The latter are called “invariants”. Homeomorphisms can often be found by means of explicit maps, as shown in the next diagram.

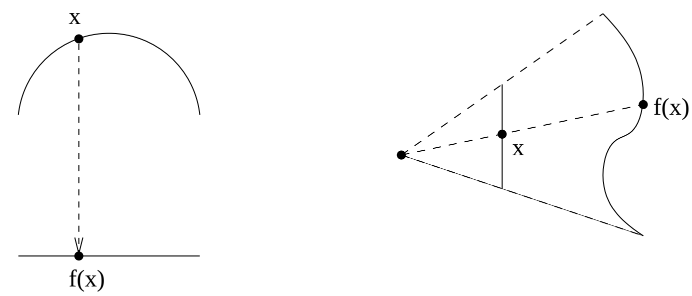


Figure 3: Two homeomorphisms

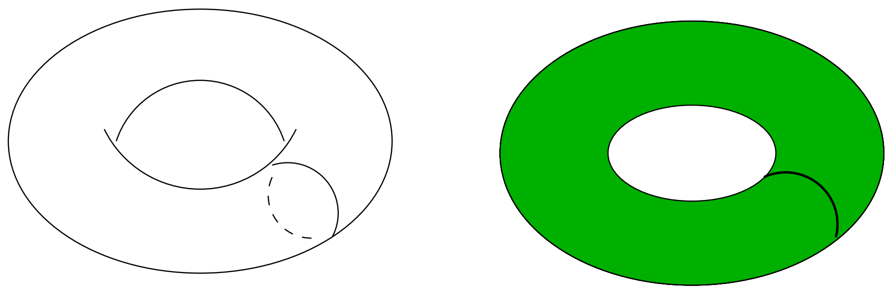


Figure 4: The Torus and the Solid Torus

How can we decide that two spaces are not homeomorphic? We can say that $S^1 \ncong S^2$ because dimension is an invariant, and they have different dimensions. We can say that $S^1 \ncong \mathbb{R}$, even though both have dimension 1, because S^1 is compact and $\mathbb{R}$ is not, where the topological invariant "compactness" is an elementary one but would take too long to give the full definition here. We can say that the real line $\mathbb{R}$ and the disjoint union of two real lines (imagine two parallel lines) are not homeomorphic, even though both are 1-dimensional and both are not compact, since the first is a connected space and the second is not connected (connectivity is another topological invariant which is easy to understand intuitively).

From given spaces it is useful to be able to construct new ones. One important way to do this is called the "product" or "cartesian product" $X \times Y$ of the two spaces X, Y. An example of this is the familiar euclidean plane $\mathbb{R}^2$, it is simply the product $\mathbb{R} \times \mathbb{R}$. If X, Y have dimensions m, n respectively, then $X \times Y$ will have dimension $m + n$. Another simple example is the torus $S^1 \times S^1$ (see Fig. 4). Be careful, this is a 2-dimensional space and is just the surface, for example the "skin" of the donut. The solid donut (the "meat") would be given by $S^1 \times D^2$, also illustrated below (Fig. 4).

Now let's try a subtler question. It "looks" intuitively that S^2 and the torus $S^1 \times S^1$ are not homeomorphic. But both are 2-dimensional, both are compact, and both are connected. How can they be distinguished? There are several ways topologists have found to do this, and we will mention two of these invariants here: the Euler characteristic and the fundamental group.

For the first invariant, the Euler characteristic, one simply triangulates the space (in this case the surface S^2 or $S^1 \times S^1$) and counts the number of vertices V, edges E, and faces (triangles) F. Then the Euler characteristic χ is defined to be $\chi = V - E + F$. For example we may triangulate S^2 as the surface of a tetrahedron, as shown in the next figure. Triangulations also occur in computer applications where smooth surfaces must be approximated, such as the surface of an auto.

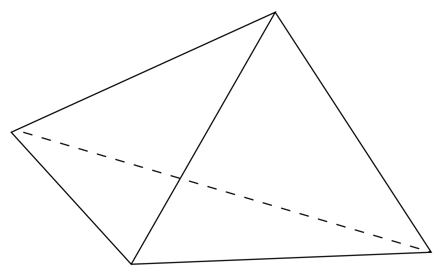


Figure 5: A Triangulation of S^2

We clearly get $\chi(S^2) = 4 - 6 + 4 = 2$, which is Euler's famous formula for the sphere : $V - E + F = 2$. It is of course crucial to know that this number χ is truly an invariant of the space, that is it will be the same no matter which triangulation is chosen for the given space. This is what the next theorem states, and a slightly unconventional proof is outlined. The theorem is for closed surfaces, by which is meant a surface that is compact, and has no boundary (such as S^2, $S^1 \times S^1$).

Theorem 1 *For any closed surface Σ, its Euler characteristic $\chi(\Sigma)$ is independent of the triangulation of Σ.*

Proof outline. We will assume the following fact : any two triangulations can be related by some finite sequence of the three simple "moves" depicted in Fig. 6.

Notice that for move (1) the net change in χ is given by $\Delta(\chi) = 1 - 3 + 2 = 0$ (similarly for move (2), the inverse of (1)), while for move (3) $\Delta(\chi) = 0 - 0 + 0 = 0$. Thus the Euler characteristic χ remains unchanged. ■

With a little effort the reader can verify that $\chi(S^1 \times S^1) = 0$. This proves the desired result, $S^2 \ncong S^1 \times S^1$, since the two spaces have different Euler characteristics.

The second method involves associating an algebraic object – a group – to each space, that is an invariant of the space. This group is called the fundamental group $\pi_1(X)$ of the space X, and it has to do with the loops in the space, which start and terminate at a fixed point called the base point of X. Two loops are considered equivalent (called homotopic) if one can be continuously deformed into the other, and loops are "multiplied" by simply following the first by the second. The fundamental group is also called the Poincaré group after its discoverer, the great French mathematician Henri Poincaré. Specialists in French history will know that there was a President

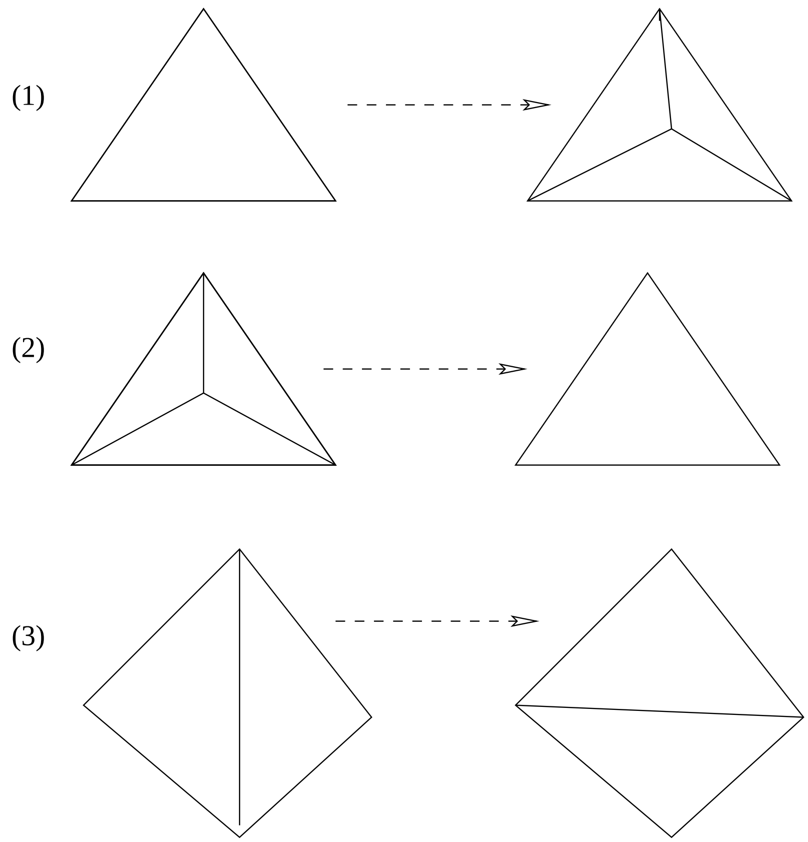


Figure 6: Basic Triangulation Moves

of France named Raymond Poincaré. Today Raymond Poincaré is probably remembered mostly because of his cousin Henri's mathematical fame.

For S^2, it is intuitively clear that any loop can be continuously shrunk to a point, thus its fundamental group is the trivial group of just a single element written $\pi_1(S^2) = \{e\}$. It turns out that the fundamental group of the torus is non-trivial, in the language of group theory one writes $\pi_1(S^1 \times S^1) \approx \mathbb{Z} \oplus \mathbb{Z}$. So we have found a second proof that these spaces are not homeomorphic, since they have different fundamental groups. In the following diagram some typical loops on the torus are shown. Loop a is homotopically trivial (can be shrunk to a point), loop b (a longitude) and loop c (a meridian) are not homotopically trivial.

A useful method of constructing new spaces from old ones is cutting and pasting. The next diagram shows how to obtain a two-handle surface (also called the "dogbone" space) by cutting and pasting two tori together. Similarly one can create a surface Σ_g with g handles, also illustrated below. Notice that $\Sigma_0 = S^2$, $\Sigma_1 = S^1 \times S^1$.

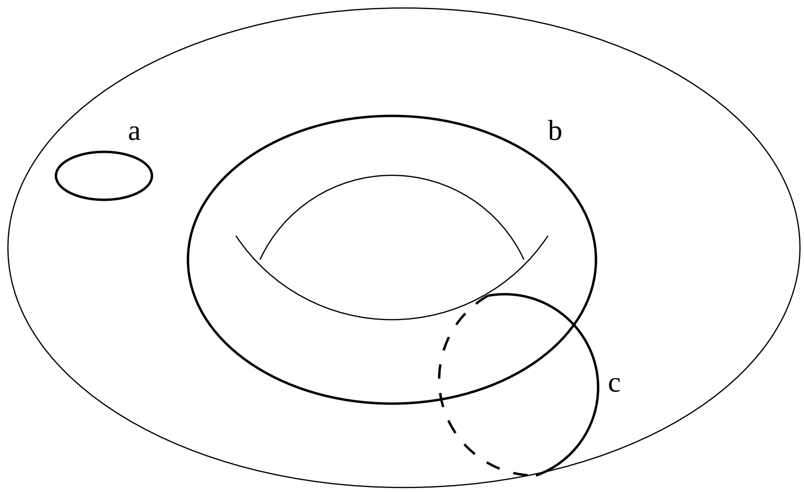


Figure 7: Loops on a Torus

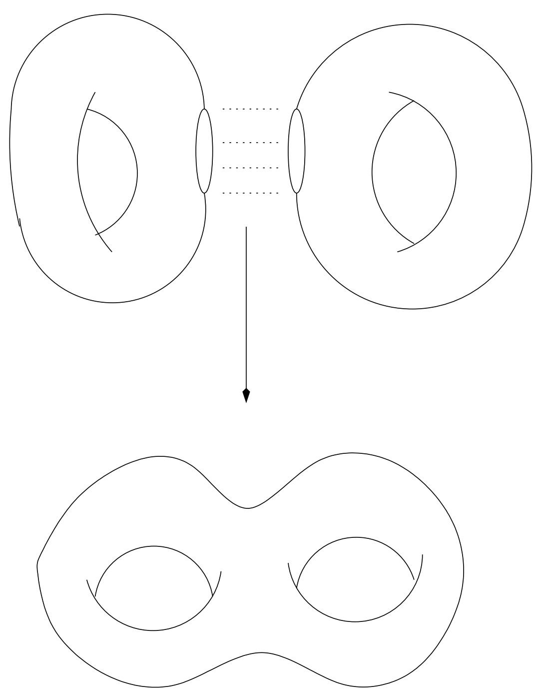


Figure 8: Connected Sum of two Tori

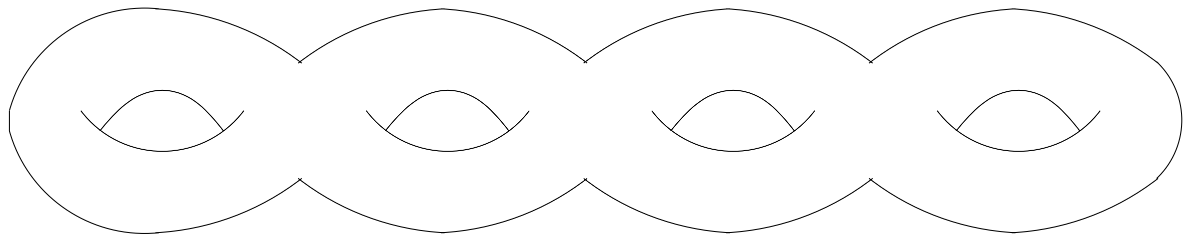


Figure 9: A Surface of Genus 4

Theorem 2 *We have:* $\Sigma_g \cong \Sigma_h$ *if and only if* $g = h$.

Proof. It is not hard to show that $\chi(\Sigma_g) = 2 - 2g$, and clearly $2 - 2g = 2 - 2h$ if and only if $g = h$. ■

Remark: Another proof can be obtained by using the fundamental group.

So far we have mainly talked about 1-dimensional and 2-dimensional spaces. In topology one deals with spaces of arbitrary dimension. One would expect intuitively that the subject gets more difficult and complicated the higher the dimension is, but one amazing surprise in topology is that this is not so. In fact the dimensions 3 and 4 are the most difficult to deal with. Somehow, in dimensions greater than 4, there is more "room" to move things around and this makes the topology simpler. We shall now turn to some 3-dimensional spaces, also called 3-manifolds.

The simplest example of a 3-dimensional space is probably the 3-sphere

$$S^3 = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

Just like the 2-sphere S^2, every loop on S^3 is contractible to a point, i.e., $\pi_1(S^3) = \{e\}$ is trivial. The converse of this is one of the most outstanding unsolved problems of mathematics, called the Poincaré Conjecture. It asserts that every closed connected 3-manifold with trivial fundamental group must be homeomorphic to S^3. There is a \$1,000,000 prize for the solution of the Poincaré Conjecture, offered by the Clay Institute in Boston.

Another way to create interesting 3-dimensional spaces will be shown by first considering the following construction of the torus. As Fig. 10 shows, we start with the disjoint union $S^1 \times D^1 \sqcup S^1 \times D^1$ of two annuli. Now glue together the inside boundary circles as suggested in Fig. 10 (a with a') and similarly for the outside boundary circles (b with b'). The result is clearly the torus.

Similarly we can start with two solid tori (3-dimensional annuli) $S^1 \times D^2 \sqcup S^1 \times D^2$. The boundary of each is a torus, and we glue these two bounding tori together. However, now there are many possible ways of gluing since the torus has many self-homeomorphisms. In fact each self-homeomorphism determines two relatively prime positive integers p, q, in such a way that the meridian of the first torus becomes a (p, q) torus knot in the second torus. This is illustrated in the next figure for the case $(p, q) = (3, 2)$, notice that the curve winds around the torus 3 times

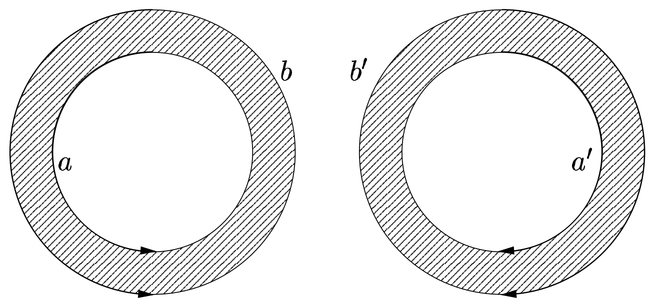


Figure 10: Construction of a Torus From Two Annuli

longitudinally and 2 times around the meridian (the reader may like to try to visualize how the knot pictured here is in fact the same as the familiar trefoil knot).

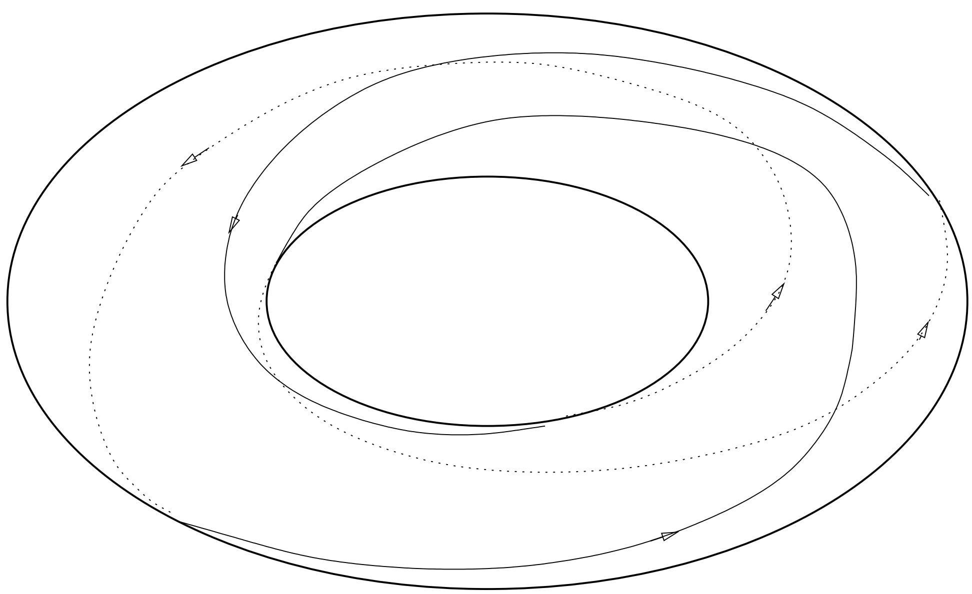


Figure 11: A (3,2) Torus Knot

If we glue the two solid tori together along their torus boundary via the (p, q) homeomorphism, the result is a 3-dimensional space that is a closed 3-manifold called the lens space $L(p, q)$. Now let us ask whether we can tell apart these lens spaces, up to homeomorphism. The Euler characteristic will give us no information at all, since it equals 0 for any closed 3-manifold. But the fundamental group does help, indeed it can be shown that $\pi_1(L(p, q)) \approx \mathbb{Z}/p\mathbb{Z}$, the integers modulo p. Thus the lens space determines p uniquely, but what about q? In 1935 the German mathematician Kurt Reidemeister proved the following theorem, which completely answers this question.

Theorem 3 *Two lens spaces* $L(p,q)$, $L(p',q')$ *are homeomorphic if and only if* $p = p'$ *and either* $p|(q \pm q')$ *or* $p|(qq' \pm 1)$.

Examples: (a) $L(7,1) \ncong L(7,2)$ since none of 1 ± 2, $1 \cdot 2 \pm 1$ are not divisible by 7.
(b) $L(7,2) \cong L(7,3)$, since $7|(2 \cdot 3 + 1)$.

The new invariant Reidemeister discovered in 1935 (cf. [14]) to prove this theorem is called Reidemeister torsion. He defined this by topological means using triangulations, the incidence matrix for the odd dimensional simplexes (the columns) versus the even dimensional simplexes (the rows), and the determinant of this (square) matrix. W. Franz [1], also in 1935, generalized this to higher dimensions, and further important work was done by J.H.C. Whitehead [22]. For expositions, see [10] and [21] where the reader will find the proofs of many theorems stated in these lectures.

This concludes our first lecture. In the next lecture the algebraic machinery involved in the Reidemeister torsion will be presented.

Lecture 2. Torsion of Chain Complexes

Let $\mathbb{F}$ be a field of arbitrary characteristic and consider the chain complex (which is referred to simply as a complex)

$$\mathcal{C} = \quad (C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \dots \xrightarrow{\partial_0} C_0),$$

where each C_i is a finite dimensional vector space over $\mathbb{F}$, and ∂_i is a vector space morphism satisfying $\partial_{i-1} \circ \partial_i = 0$, for all $i = 1, \dots, m-1$.

A somewhat more general definition is often given, where the sequence is allowed to extend arbitrarily in both directions. The chain complex $\mathcal{C}$ defined above can also be viewed in this way, by simply specifying that $C_{m+1} = C_{m+2} = \dots = 0 = C_{-1} = C_{-2} = \dots$, and consequently $\partial_m = \partial_{m+1} = \dots = 0 = \partial_{-1} = \partial_{-2} = \dots$. The definition implies that $\mathrm{Ker}\partial_i \supseteq \mathrm{Im}\partial_{i+1}$, and this leads to the definition of the homology of a complex.

Definition 2.1. The i-th *homology* of the complex C is $H_i(\mathcal{C}) = \mathrm{Ker}\ \partial_{i-1}/\mathrm{Im}\ \partial_i$.

As a subquotient of vector spaces, $H_i(C)$ is a vector space over $\mathbb{F}$. In particular, $H_0(C) = \mathrm{Ker}\ \partial_{-1}/\mathrm{Im}\ \partial_0 = C_0/\mathrm{Im}\ \partial_0 = \mathrm{Coker}\ \partial_0$ and $H_m(C) = \mathrm{Ker}\ \partial_{m-1}/\mathrm{Im}\ \partial_m = \mathrm{Ker}\ \partial_{m-1}$.

Definition 2.2. The complex $\mathcal{C}$ is said to be *acyclic* if $H_i(\mathcal{C}) = 0$ for all i.

The acyclic property is equivalent to the relation $\mathrm{Ker}\ \partial_{i-1} = \mathrm{Im}\partial_i$ for each i, which means that the sequence is exact.

Definition 2.3. The chain complex $\mathcal{C}$ is said to be *based* if each vector space C_i is equipped with a distinguished (ordered) basis, which will be denoted by c_i.

The first step will be to define the torsion $\tau(\mathcal{C}) \in \mathbb{F}^* = \mathbb{F} \setminus \{0\}$ of a based acyclic chain complex $\mathcal{C}$. In order to do this, some notation will first be given. Let V be a finite dimensional vector space, of dimension k, over $\mathbb{F}$. Choose two (ordered) bases $v = (v_1, \dots, v_k)$ and $u = (u_1, \dots, u_k)$ of V. Let (a_{ij}), where $0 \le i, j \le k$, denote the transition matrix from the basis u to the basis v. Furthermore let

$$[v/u] = \det\ (a_{ij}) \in \mathbb{F}^* = \mathbb{F} \setminus 0\,.$$

It is clear that

(i) $[v/v] = 1$, and

(ii) if w is a third basis of V, then $[v/w] = [v/u] \cdot [u/w]$.

Two bases u and v of V are called *equivalent* (written $u \cong v$) if $[v/u] = 1$.

Let C be an acyclic complex. Let $B_i = \mathrm{Im}\ \partial_i \subseteq C_i$. Since $\mathcal{C}$ is acyclic, $B_i = \mathrm{Ker}\ \partial_{i-1}$. It follows that $C_i/B_i = \mathrm{Im}\ \partial_{i-1} = B_{i-1}$. This is equivalent to the statement that the following sequence is exact:

$$0 \to B_i \hookrightarrow C_i \twoheadrightarrow B_{i-1} \to 0.$$

For each $i = 0, 1, \ldots, m-1$, choose a basis b_i for B_i, and let $\widetilde{b_{i-1}} \subset C_i$ be a lift of b_{i-1} to C_i (of course many choices will be possible for $\tilde{b}_{i-1}$). From the above short exact sequence it is clear that $b_i \cup \widetilde{b_{i-1}}$ forms a basis for C_i. Denote this basis by $b_i\tilde{b}_{i-1}$ and as above let $[b_i\tilde{b}_{i-1}/c_i]$ denote the determinant of the transition matrix for the change of basis from c_i to $b_i\tilde{b}_{i-1}$.

Lemma 1 *The element $[b_i\tilde{b}_{i-1}/c_i]$ is non-zero in $\mathbb{F}$ and does not depend on the choice of the lift $\tilde{b}_{i-1}$ of b_{i-1}.*

Proof. The first assertion is clear. Let $\tilde{b}'_{i-1}$ be another choice of lift of b_{i-1}. Then $\tilde{b}'_{i-1} - \tilde{b}_{i-1} \subset \ker \partial_{i-1} = B_i$. Hence $[b_i\tilde{b}'_{i-1}/b_i\tilde{b}_{i-1}] = 1$ (the transition matrix is triangular with 1's on the diagonal). Use property (ii) to conclude that $[b_i\tilde{b}'_{i-1}/c_i] = [b_i\tilde{b}_{i-1}/c_i]$. ■

Definition 2.4. The torsion of a based acyclic complex $\mathcal{C}$ is defined to be:

$$\tau(\mathcal{C}) = \prod_{i=0}^{m} [b_i\tilde{b}_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}^* .$$

Lemma 2 *The torsion $\tau(\mathcal{C})$ of the complex $\mathcal{C}$ is independent of the choice of the bases b_i.*

In order to prove this lemma, it is sufficient to show that

$$[b_i\tilde{b}_{i-1}/c_i]^{(-1)^{i+1}} [b_{i+1}\tilde{b}_i/c_{i+1}]^{(-1)^{i+2}}$$

is independent of the choice of b_i. To see this, choose another basis b'_i of B_i and use property (ii).

Remark. The torsion $\tau(\mathcal{C})$ does depend on the distinguished basis c_i of C_i. For, if $\mathcal{C}'$ is the same acyclic chain complex based by $c' = (c'_0, \ldots, c'_m)$, then

$$\tau(\mathcal{C}') = \tau(\mathcal{C}) \cdot \prod_{i=0}^{m} [c_i/c'_i]^{(-1)^{i+1}} .$$

Example. Let $m = 1$. Then

$$\mathcal{C} = (0 \to C_1 \xrightarrow{\partial_0} C_0 \to 0) .$$

By acyclicity ∂_0 is an isomorphism. As above, let c_0, c_1 be the distinguished bases, which in this case have the same number (say n) of elements. Then ∂_0 is given by an $n \times n$ matrix, say A, with respect to these bases, and $\tau(\mathcal{C}) = (\det A)^{-1}$.

Proof. Take $b_0 = c_0$, $b_1 = \emptyset$. It follows that $[b_0/c_0] = [c_0/c_0] = 1$, and $[b_1\tilde{b}_0/c_1] = [\tilde{b}_0/c_1] = \det\,(A^{-1}) = (\det\, A)^{-1}$. Thus

$$\tau(\mathcal{C}) = [b_0/c_0]^{-1} \cdot [b_1\tilde{b}_0/c_1] = [\tilde{b}_0/c_1] = (\det\, A)^{-1} ,$$

as asserted. ∎

Since the torsion is defined using determinants, it is not too surprising that it enjoys some properties similar to those of determinants. For example the next theorem has some analogy to the property

$$\det \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix} = \det(A_1) \cdot \det(A_2),$$

where A_1 and A_2 are square matrices.

Theorem 4 (Multiplication of torsions) *For any short exact sequence*

$$0 \rightarrowtail \mathcal{C}' \to \mathcal{C} \twoheadrightarrow \mathcal{C}'' \to 0\,,$$

of acyclic based chain complexes satisfying $[c_i/c'_i\ \tilde{c}''_i] = 1$, *for all* $i = 0, \ldots, m$,

$$\tau(\mathcal{C}) = \pm\tau(\mathcal{C}') \cdot \tau(\mathcal{C}'')\,.$$

Note that the condition $[c_i/c'_i\ \tilde{c}''_i] = 1$ is independent of the choice of the lifts $\tilde{c}''_i$. The proof of the theorem consists first in observing that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B'_i & \longrightarrow & B_i & \longrightarrow & B''_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C'_i & \longrightarrow & C_i & \longrightarrow & C''_i & \longrightarrow & 0 \end{array}$$

is commutative; the result follows by comparing two different bases of C_i: one is obtained from bases of C'_i and C''_i respectively, another one from bases of B'_i, B''_i, B'_{i-1}, B''_{i-1} respectively. ∎

To compute the torsion, it is convenient to introduce the following definition. Let $\mathcal{C}$ denote a fixed acyclic and based chain complex. The map $\partial_i\colon C_{i+1} \to C_i$ is then given by a matrix $A_i = (a^i_{jk})$, $j = 1, \ldots, \dim\ C_{i+1}$, $k = 1, \ldots, \dim\ C_i$.

Definition 2.5. Consider a collection of sets $\alpha = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ where $\alpha_i \subseteq \{1, 2, 3, \ldots, \dim\ C_i\}$. Define $S_i = S_i(\alpha)$ to be the submatrix of A_i formed by the entries a^i_{jk} of A_i such that $j \in \alpha_{i+1}$ $(0 \le i \le m-1)$ and $k \notin \alpha_i$. The collection of sets α is called a *τ-chain* if

(i) $\alpha_0 = \phi$,

(ii) each $S_i(\alpha)$ is a square matrix.

The τ-chain α is said to be *non-degenerate* if $\det S_i \neq 0$ for all even i.

Example. For $\dim C_{i+1} = 3$, $\dim C_i = 5$, $\alpha_i = \{1, 3, 5\}$, $\alpha_{i+1} = \{1, 3\}$. The submatrix $S_i = S_i(\alpha)$ is the 2×2 submatrix below

$$\begin{array}{|ccccc|} \hline a_{11} & \boxed{a_{12}} & a_{13} & \boxed{a_{14}} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & \boxed{a_{32}} & a_{33} & \boxed{a_{34}} & a_{35} \\ \hline \end{array}\,.$$

Theorem 5 *If α is a non-degenerate τ-chain then* $\det S_i \neq 0$ *for all i and*

$$\tau(\mathcal{C}) = \pm \prod_{i=0}^{m-1} (\det S_i)^{(-1)^{i+1}}.$$

Remark. The sign $\pm$ referred to in the previous theorem is equal to $(-1)^N$ where

$$N = \sum_{i=0}^{m} \#\{(x,y) \colon x < y, x \in \alpha_i, y \in \{1, 2, \ldots, \dim C_i\} \setminus \alpha_i\}$$

Theorem 6 *A based chain complex is acyclic if and only if it has a non-degenerate τ-chain.*

In particular, the torsion of acyclic complexes can always be computed using non-degenerate τ-chains.

More generally, chain complexes can be defined over any associative ring with unit. Let $\mathcal{C}$ be a chain complex over a commutative, Noetherian, unique factorization domain R. Assume that $\mathcal{C}$ is based and that rank $H_*(\mathcal{C}) = 0$. Let $\tilde{R}$ denote the field of quotients of R. (Observe that for any R-module M, rank $(M) = \dim_{\tilde{R}}(\tilde{R} \otimes_R M)$.) Then the chain complex $\tilde{R} \otimes_R \mathcal{C}$ over $\tilde{R}$ is based and acyclic and we can consider ist torsion $\tau(\tilde{R} \otimes_R C) \in \tilde{R}$. We now compute this torsion in homological terms, at least up to multiplication by invertible elements of R.

Definition 2.6. Let M be a finitely generated R-module. A *presentation* of M is an exact sequence

$$R^m \xrightarrow{f} R^n \to M \to 0.$$

Let $A = (a_{jk})_{\substack{1 \le j \le m \\ 1 \le k \le n}}$ denote the matrix of the homomorphism $f \colon R^m \to R^n$ with respect to the standard bases in R^m and R^n. The j-th row corresponds to the image of the j-th element of the basis of R^m. Therefore the columns correspond to the generators of R^n and the rows correspond to the relations between the generators. Conversely, each such $m \times n$ matrix describes a presentation of a finitely generated R-module.

If A is a presentation matrix of the finitely generated R-module M, then for $k \geq 0$, the k-th *elementary ideal* of M is the ideal $E_k(M) = E_k(A) \subset R$ generated by the $(n-k) \times (n-k)$ minors of A. If $n - k \leq 0$ then by definition $E_k(M) = R$. If $n - k > m$, then $E_k(M) = 0$. Note that $E_k(M) \subseteq E_{k+1}(M)$ for $k = 0, 1, 2, \ldots$.

Lemma 3 *The ideals $E_k(M)$ do not depend on the choice of A.*

Define $\Delta_k(M) = \gcd\,(E_k(M)) \in R$. Thus, $\Delta_k(M)$ is a generator of the smallest principal ideal containing $E_k(M)$ and $\Delta_{k+1}(M) | \Delta_k(M)$ for all $k = 0, 1, \ldots$. The element $\Delta_0(M)$ is called the *order* of M, also denoted $\mathrm{ord}(M)$.

Theorem 7 *Consider the chain complex $\tilde{R} \otimes_R \mathcal{C}$. Then*

$$\tau(\tilde{R} \otimes_R \mathcal{C}) = \prod_{i=0}^{m} (\text{ord } H_i(\mathcal{C}))^{(-1)^{i+1}} .$$

Suppose now that $\mathcal{C}$ is a not necessarily acyclic based chain complex over $\mathbb{F}$ and that $H_i(C) = \text{Ker } \partial_{i-1}/\text{Im } \partial_i$ is also based. Set $Z_i = \text{Ker } (\partial_{i-1}\colon C_i \to C_{i-1})$. Then $0 \subseteq B_i \subseteq Z_i \subseteq C_i$ and

$$Z_i/B_i = H_i(\mathcal{C}), \qquad C_i/Z_i \simeq B_{i-1} .$$

Let c_i and h_i be the distinguished bases of C_i and $H_i(\mathcal{C})$, respectively. Choose any basis b_i of B_i. Then b_i, h_i and $\tilde{b}_{i-1}$ (lift of b_{i-1} in C_i) form a basis $b_i h_i \tilde{b}_{i-1}$ of C_i.

Definition 2.7. The torsion of a chain complex $\mathcal{C}$, with based homology, is defined by

$$\tau(\mathcal{C}) = \prod_{i=0}^{m} [b_i h_i \tilde{b}_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}^* .$$

This generalizes the original definition of $\tau(\mathcal{C})$ in the acyclic case. Once again the torsion does not depend on the choice of the b_i's. However, it does depend on the choice of c_i and h_i.

Lecture 3. The Torsion of CW-Complexes

In this third lecture, we turn to topology and define the torsion of a CW-complex.

Definition 3.1. A *finite CW-complex* X is a Hausdorff space which is the union of a finite number of disjoint subspaces e_α called cells satisfying the following conditions:

(1) To each cell is associated an integer $k \geq 0$ (the *dimension* of the cell). The union X^n of all k-cells for $k \leq n$ is called the *n-skeleton*. We have $X = \cup_{n\geq 0} X^n$.

(2) If e_α^k is a k-cell, there is a characteristic map $\chi_\alpha : (B^k, S^{k-1}) \to (X, X^{k-1})$ such that $\chi_\alpha|_{B^k - S^{k-1}}$ is a homeomorphism from $B^k - S^{k-1}$ onto e_α^k.

The smallest integer n such that $X^n = X$ is called the *dimension* of X and is denoted $\dim X$.

Remark. Note that a finite CW-complex is compact since it is covered by a finite number of compact sets $\{\chi_\alpha(B^n)\}$.

Example. See Figure 12. The circle S^1 is a finite CW-complex with two cells e^0, $e^1 = S^1 - e^0$.

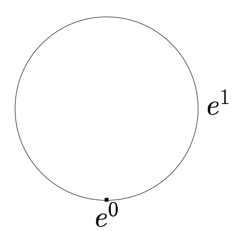


Figure 12: A cell decomposition of the unit circle.

Let X be a finite connected CW-complex with a fixed base point x (a 0-cell). Let $\pi = \pi_1(X, x)$ be the fundamental group. Set $H = H_1(X; \mathbb{Z}) = \pi/[\pi, \pi]$. Let $\mathbb{Z}[H]$ be the group ring of H, that is the set of formal linear combinations of elements of H with coefficients in $\mathbb{Z}$ (it has a natural commutative ring structure with unit). Fix a ring homomorphism $\varphi : \mathbb{Z}[H] \to \mathbb{F}$, where $\mathbb{F}$ is a field. We shall present below the construction of Reidemeister and Franz (1935) of the torsion associated to φ and X.

Consider the maximal abelian covering $\tilde{X}$ of X. It is the regular covering of X associated to the kernel of the homomorphism $\pi \to H$, i.e., to the subgroup $[\pi, \pi] \trianglelefteq \pi$. The group $H = \pi/[\pi, \pi]$ acts freely on $\tilde{X}$ and $\tilde{X}/H = X$. Note that the CW-structure in X lifts to $\tilde{X}$: each cell can be lifted (in many different ways, according to the action of H) to $\tilde{X}$. The free action of H on $\tilde{X}$ gives rise to an action of $\mathbb{Z}[H]$ on the cellular chain complex $C_*(\tilde{X})$. Thus $C_*(\tilde{X})$ becomes a (free) $\mathbb{Z}[H]$-module.

Example. $X = S^1$, $\tilde{X} = \mathbb{R}$ and the covering map is the exponential map $\mathbb{R} \to S^1, x \mapsto \exp(2\pi i x)$, $H = \mathbb{Z}$ acts by translations on $\mathbb{R}$. Let t denote a generator of H. Then $\mathbb{Z}[H] = \mathbb{Z}[t, t^{-1}]$. According to the choice (see Fig. 13) of lifts $\tilde{e}^0$ and $\tilde{e}^1$ of cells, we have: $C_0(\mathbb{R}) = \mathbb{Z}[t, t^{-1}]\tilde{e}^0$ and $C_1(\mathbb{R}) = \mathbb{Z}[t, t^{-1}]\tilde{e}^1$. For an appropriate choice of orientation of $\tilde{e}^1$, the boundary map is given by $\partial \tilde{e}^1 = (t-1)\tilde{e}^0$.

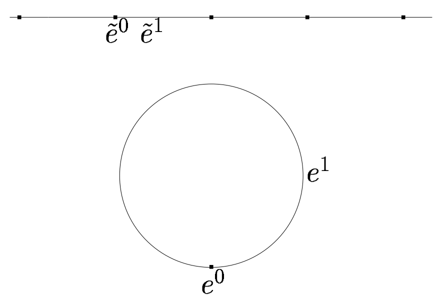


Figure 13: The maximal abelian covering of S^1 with lift of the cellular decomposition.

Note that $\mathbb{Z}[H]$ acts both on $\mathbb{F}$ and on $C_*(\tilde{X})$:

- $\mathbb{Z}[H]$ acts on $\mathbb{F}$ via the homomorphism $\varphi : \mathbb{Z}[H] \to \mathbb{F}$: $x \cdot f = \varphi(x)f$, $x \in \mathbb{Z}[H], f \in \mathbb{F}$.
- $\mathbb{Z}[H]$ acts on $C_*(\tilde{X})$ via covering transformations.

Hence one can define $C_*^\varphi(X) = \mathbb{F} \otimes_{\mathbb{Z}[H]} C_*(\tilde{X})$. Let $i \in \mathbb{N}$. Define the *i-dimensional twisted homology of X* by $H_i^\varphi(X) = H_i(C_*^\varphi(X))$. It is a vector space over F and is a homotopy invariant of X. Torsion appears as a "secondary" invariant of X, namely when $H_*(C_*^\varphi(X)) = 0$. From now on, we assume that $H_*(C_*^\varphi(X)) = 0$, i.e., $C_*^\varphi(X)$ is acyclic.

Definition 3.2. A *fundamental family of cells* in $\tilde{X}$ is a family $\mathcal{F}$ of cells of $\tilde{X}$ such that over each cell of X lies exactly one cell of this family.

Example. The previous example yields a fundamental family of cells: $\mathcal{F} = (\tilde{e}^0, \tilde{e}^1)$.

If each cell is oriented and the cells in each dimension are ordered, then a fundamental family $\mathcal{F}$ yields a basis for $C_*^\varphi(X)$. Consider the element

$$\tau^\varphi(X) = \tau(C_*^\varphi(X), \mathcal{F}) \in \mathbb{F}^* = \mathbb{F} - \{0\}.$$

Let us see how a different choice of a fundamental family of cells affects $\tau^\varphi(X)$. Any two lifts of a cell e_i to a cell in $\tilde{X}$ are related by an element $h \in H$. If a cell $\tilde{e}_i$ over e_i in the fundamental family is replaced by another one, say $h\tilde{e}_i$, the torsion changes by a factor $\varphi(h)^{(-1)^{\dim e_i}}$. If the orientation of a cell is inverted or if the order of cells is changed, then the torsion is multiplied by ± 1. As a conclusion,

$$\tau^{\varphi}(X) \in \mathbb{F}^* / \pm \varphi(H). \tag{1}$$

In other words, the torsion is well-defined as an element in $\mathbb{F}^* / \pm \varphi(H)$. Note that $\varphi(H) \subset \mathbb{F}^*$ since H lies in the group of invertible elements of $\mathbb{Z}[H]$.

Theorem 8 1). *$\tau^{\varphi}(X)$ is invariant under cell subdvisions.* 2). *$\tau^{\varphi}(X)$ is invariant under homeomorphisms.*

Remark. 1) allows to define torsions for PL-manifolds (any two PL-triangulations of a PL-manifold have a common subdivision) and for smooth manifolds (using C^1-triangulations). 2) implies that the torsion is not sensitive to non-equivalent PL or smooth structures on manifolds.

Examples. 1) $X = S^1$, $F = \mathbb{Q}(t)$ (field of rational fractions). Let $\varphi : \mathbb{Z}[H] = Z[t^{\pm 1}] \to \mathbb{Q}(t)$ be the natural inclusion. Let us compute $\tau^{\varphi}(X)$. Using the fundamental family $\tilde{e}^0, \tilde{e}^1$ described in Fig. 13, we have $C_0(\mathbb{R}) = \mathbb{Z}[t^{\pm 1}]\tilde{e}^0$, $C_1(\mathbb{R}) = \mathbb{Z}[t^{\pm 1}]\tilde{e}^0$ and $\partial \tilde{e}^1 = (t-1)\tilde{e}^0$. The resulting twisted chain complex $C_*^{\varphi}(\mathbb{R})$ over $\mathbb{Q}(t)$ is $\mathbb{Q}(t)\tilde{e}^1 \overset{\partial}{\to} \mathbb{Q}(t)\tilde{e}^0$ with the boundary map ∂ defined by $\partial \tilde{e}^1 = \varphi(t-1)\tilde{e}^0 = (t-1)\tilde{e}^0$. Since ∂ is an isomorphism, the complex $C_*^{\varphi}(\mathbb{R})$ is acyclic. The determinant of ∂ (with respect to the bases $\tilde{e}^0$ and $\tilde{e}^1$) is $t-1$. Hence $\tau^{\varphi}(\mathbb{R}) = \frac{1}{t-1}$.

2) Lens spaces. Let $p \geq 2$ and let $q_1, \dots, q_n$ be integers coprime to p. The group $C_p = \{\xi \in \mathbb{C},\ \xi^p = 1\}$ of complex p-th roots of unity acts freely on the oriented sphere $S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n,\ |z_1|^2 + \cdots + |z_n|^2 = 1\}$ by

$$\xi \cdot (z_1, \dots, z_n) = (\xi^{q_1} z_1, \xi^{q_2} z_2, \dots, \xi^{q_n} z_n).$$

Consider the orbit space $L(p; q_1, \dots, q_n) = S^{2n-1}/C_p$. It is a closed oriented $(2n-1)$-manifold, called a *lens space*. Note that it does not depend on the particular order of the q_i's. Since S^{2n-1} is simply-connected, the fundamental group of $L(p; q_1, \dots, q_n)$ is C_p (in particular it is abelian). The full classification of lens spaces is due to Reidemeister for $n = 2$ and to Franz for $n \geq 3$. It is given in the following theorem.

Theorem 9 *$L(p; q_1, \dots q_n) \cong L(p; q_1', \dots q_n')$ if and only if there exists $r \in \mathbb{Z}$ such that $(r, p) = 1$ and $\{rq_i \pmod p\} = \{\pm q_i' \pmod p\}$.*

Reidemeister's and Franz's proof are based on torsion. Set $H = H_1(L(p; q_1, \dots, q_n)) = C_p$. For $\xi \in C_p$, denote by φ_ξ the homomorphism $\mathbb{Z}[H] = \mathbb{Z}[t^{\pm 1}]/\langle t^p = 1 \rangle \to \mathbb{C}, t \mapsto \xi$. The torsion of $L(p; q_1, \dots q_n)$ is computed by explicitly choosing a cell decomposition of $L(p; q_1, \dots q_n)$:

$$\tau^{\varphi_\xi}(L(p; q_1, \dots q_n)) = \frac{1}{(\xi^{q_1} - 1)(\xi^{q_2} - 1) \cdots (\xi^{q_n} - 1)} \in \mathbb{C}/ \pm C_p.$$

It requires some number theory to show that this formula implies the conditions stated in the theorem. Clearly, the conditions of the Theorem are sufficient for two lens spaces $L(p; q_1, \dots q_n)$ and $L(p; q_1', \dots q_n')$ to be homeomorphic. Hence the torsion determines the homeomorphism type of lens spaces. ■

Remark. Three-dimensional lens spaces (introduced in Lecture 1) are obtained as a particular case of the lens spaces defined above. With the notation of Lecture 1, $L(p, q) = L(p; q, 1) = L(p; 1, q)$.

To get rid of the indeterminacy in the definition of the torsion, one can equip the CW-complex X with two extra structures:

- a homology orientation,
- an Euler structure.

The homology orientation will eliminate the sign indeterminacy of the torsion, whereas the Euler structure will eliminate the $\varphi(H)$ indeterminacy in (1).

First a refinement of the torsion of chain complexes needs to be introduced. (The torsion of a general (non necessarily acyclic) chain complex was defined in Lecture 2.) Let $\mathcal{C} = (0 \to C_m \overset{\partial_{m-1}}{\longrightarrow} C_{m-1} \overset{\partial_{m-2}}{\longrightarrow} \dots \overset{\partial_0}{\longrightarrow} C_0 \to 0)$ be a based chain complex over a field $\mathbb{F}$. Set $\beta_i(\mathcal{C}) = \sum_{k=0}^{i} \dim H_k(\mathcal{C}) \pmod 2$ and $\gamma_i(C) = \sum_{k=0}^{i} \dim C_k \pmod 2$. Define a weight $N(\mathcal{C})$ by

$$N(\mathcal{C}) = \sum_{i=0}^{m} \beta_i(\mathcal{C})\gamma_i(\mathcal{C}) \in \mathbb{Z}_2.$$

Let c_i be a basis of C_i, b_i a basis of $B_i = \text{Im } \partial_i$ and h_i a basis of $H_i(\mathcal{C})$, $i = 0, \dots, m$. Define

$$\check{\tau}(\mathcal{C}, c, h) = (-1)^{N(\mathcal{C})}\tau(\mathcal{C}) = (-1)^{N(\mathcal{C})} \prod_{i=0}^{m} [b_i h_i b_{i-1}/c_i]^{(-1)^{i+1}} \in \mathbb{F}^* . \tag{2}$$

In the case $\mathcal{C}$ is acyclic, $N(\mathcal{C}) = 0$ and $\check{\tau}(\mathcal{C}, c, h) = \tau(\mathcal{C})$. Thus the refinement introduced in (2) is relevant only in the case when $\mathcal{C}$ is not acyclic and has non trivial homology.

Definition 3.3. A *homology orientation* of a finite connected CW-complex X is an orientation of $H_*(X; \mathbb{R}) = \oplus_{i \geq 0} H_i(X; \mathbb{R})$ (as a vector space over $\mathbb{R}$).

Let X be a finite CW-complex, equipped with a homology orientation ω. Let $h = (h_0, \dots, h_{\dim X})$ be a positive basis of $H_*(X; \mathbb{R}) = \oplus_{i \geq 0} H_i(X; \mathbb{R})$, i.e., a basis in the class determined by ω. Let

$\varphi : \mathbb{Z}[H] \to \mathbb{F}$ be a fixed ring homomorphism. Let e denote an ordered and oriented collection of all cells of X. This collection determines a basis of $C_*(X;\mathbb{R})$, denoted by $e_{\mathbb{R}}$. Then $\check{\tau}(C_*(X;\mathbb{R}), e_{\mathbb{R}}, h) \in \mathbb{R}^* = \mathbb{R} - \{0\}$. Define

$$\operatorname{sign}(X, e, h) = \operatorname{sign}\left(\check{\tau}(C_*(X;\mathbb{R}), e_{\mathbb{R}}, h)\right) \in \{\pm 1\}.$$

Lift e to an ordered and oriented fundamental family $\mathcal{F}$ of cells in the maximal abelian covering $\tilde{X}$. This family yields a basis of $C_*(\tilde{X})$ over $\mathbb{Z}[H]$. Therefore we also obtain a basis $\tilde{e}$ of $C_*^{\varphi}(X) = \mathbb{F} \otimes_{\mathbb{Z}[H]} C_*(\tilde{X})$. Assuming that $H_*^{\varphi}(X) = 0$, there is a well-defined torsion $\tau(C_*^{\varphi}(X), \mathcal{F}) \in \mathbb{F}^*$.

Definition 3.4. The sign-refined torsion of X is $\tau^{\varphi}(X, \omega, \mathcal{F}) = \tau(C_*^{\varphi}(X), \mathcal{F}) \cdot \operatorname{sign}(X, e, h)$.

Theorem 10 *$\tau^{\varphi}(X, \omega, \mathcal{F})$ is independent of the order of the cells of X, their orientations and the choice of an $\omega-$positive basis h.*

Proof. A change in the order of the cells of X affects both $\tau^{\varphi}(C_*^{\varphi}(X), \mathcal{F})$ and $\check{\tau}(C_*(X;\mathbb{R}), e_{\mathbb{R}}, h)$ by the same ± 1 sign change so $\tau^{\varphi}(X, \omega, \mathcal{F})$ is globally unaffected. A similar argument holds in the case of a change of orientation of a cell. Finally, if h and h' are two ω-positive bases of $H_*(X;\mathbb{R})$, then

$$[b_i h_i' b_{i-1}/c_i] = [b_i h_i' b_{i-1}/b_i h_i b_{i-1}] \cdot [b_i h_i b_{i-1}/c_i] = [h_i'/h_i] \cdot [b_i h_i b_{i-1}/c_i].$$

If follows that

$$\check{\tau}(C_*(X;\mathbb{R}), e_{\mathbb{R}}, h') = \prod_{i \geq 0} \operatorname{sign}[h_i'/h_i] \cdot \check{\tau}(C_*(X;\mathbb{R}), e_{\mathbb{R}}, h).$$

Since both h and h' are ω-positive, $\prod_{i \geq 0} \operatorname{sign}[h_i'/h_i] = 1$. ■

Recall that a change in the fundamental family of cells in $\tilde{X}$ will affect $\tau^{\varphi}(C_*^{\varphi}(X), \mathcal{F})$ – and thus also $\tau^{\varphi}(X, \omega, \mathcal{F})$ – by a multiplicative factor $\varphi(h)$, $h \in H$. Hence from Theorem 10 we deduce

Corollary 10.1 *$\tau^{\varphi}(X, \omega) = \tau^{\varphi}(X, \omega, \mathcal{F})$ is a well-defined element in $\mathbb{F}^*/\varphi(H)$.*

Hence $\varphi(H)$ is the remaining indeterminacy. To remove it amounts to specifying a fundamental family of cells $\mathcal{F}$ in $\tilde{X}$.

Theorem 11 *$\tau^{\varphi}(X, \omega)$ is invariant under simple homotopy equivalences preserving the homology orientation.*

See [21] or [18] for a proof. Theorem 11 implies in particular that the sign-refined torsion $\tau^{\varphi}(X, \omega)$ is invariant under cellular subdivisions of X. Thus one may apply torsion to homology oriented compact PL-manifolds.

Euler structures and their relationship to spin^c-structures of 3-manifolds will be presented in the next lecture.

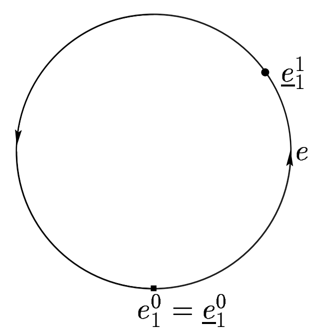
Lecture 4. Euler Structures and Refined Torsions

In this fourth lecture we will see how Euler structures can be used to remove the indeterminacy in the definition of torsion, as mentioned at the end of the previous lecture. So let us consider a CW-complex X which is finite, connected, and with vanishing Euler characteristic $\chi(X) = 0$. Any CW-complex is the disjoint union of its open cells, and we shall denote the open n-cells of X by e^n_α, $\alpha \in \mathcal{A}_n$. To further fix the notation, we choose in each open cell a base point (for that cell) $\underline{e}^n_\alpha \in e^n_\alpha$. Of course this creates many possible choices, and care will be necessary to be sure that the final outcomes are independent of these choices.

Definition 4.1. An Euler chain in X is a singular 1-chain ξ with the property

$$\partial\xi = \sum_{n\geq 0,\ \alpha\in\mathcal{A}} (-1)^n \underline{e}^n_\alpha.$$

One may write $\xi = \pm\beta_1 \pm \ldots \pm \beta_m$ for suitable arcs β_i in X. These arcs should be thought of as singular 1-simplices in X, they are not in general CW 1-simplices (which are linear combinations of the e^1_α). A simple example of an Euler chain in the circle S^1 follows. In this example S^1 has the CW structure given by a single vertex e^0_1 and a single 1-cell e^1_1. For the vertex, one must have $\underline{e}^0_1 = e^0_1$. For the 1-cell we may choose $\underline{e}^1_1$ to be any point on e^1_1. This is shown in the next diagram.



There are two obvious arcs from $\underline{e}^1_1$ to $\underline{e}^0_1$, choosing either of these gives an Euler chain ξ, since clearly $\partial\xi = \underline{e}^0_1 - \underline{e}^1_1$.

Proposition 1 *Any CW-complex X satisfying the above hypotheses admits an Euler chain.*

Proof. Using the connectivity of X (which implies arcwise connectivity since X is a CW-complex), choose for each open cell e^n_α a path β^n_α from a fixed 0-cell $\underline{e}^0_1 = e^0_1$ to $\underline{e}^n_\alpha$ (β^0_1 may be taken to be the trivial path but this makes no difference). Clearly $\partial\beta^n_\alpha = \underline{e}^n_\alpha - \underline{e}^0_1$. Let us also write $|\mathcal{A}_n| = a_n$ for the number of n-cells, then of course $\chi(X) = \sum_{n\geq 0}(-1)^n a_n$ by definition. Defining

$$\xi = \sum_{n\geq 0,\ \alpha\in\mathcal{A}_n} (-1)^n \beta^n_\alpha,$$

the following simple computation (and the hypothesis $\chi(X)=0$) completes the proof:

$$\begin{aligned} \partial\xi &= \sum_{n\geq 0,\ \alpha\in\mathcal{A}_n} (-1)^n \partial\beta_\alpha^n \\ &= \sum_{n\geq 0,\ \alpha\in\mathcal{A}_n} (-1)^n (\underline{e}_\alpha^n - \underline{e}_1^0) \\ &= \sum_{n\geq 0,\ \alpha\in\mathcal{A}_n} (-1)^n \underline{e}_\alpha^n - \sum_{n\geq 0} (-1)^n a_n \underline{e}_1^0 \\ &= \sum_{n\geq 0,\ \alpha\in\mathcal{A}_n} (-1)^n \underline{e}_\alpha^n, \end{aligned}$$

as required. ■

An Euler chain is certainly not unique, but clearly the difference $\xi-\eta$ of two Euler chains is a 1-cycle (i.e. $\partial(\xi-\eta)=0$). This difference therefore defines a homology class $[\xi-\eta]\in H_1(X)$. We define an equivalence relation $\xi\sim\eta$ if and only if $[\xi-\eta]=0\in H_1(X)$.

Definition 4.2. An Euler structure e on X is an equivalence class of Euler chains, under the above equivalence relation. The set of Euler structures on X is denoted $\mathrm{Eul}(X)$.

Writing (throughout the rest of this lecture and in subsequent lectures) $H=H_1(X)=H_1(X;\mathbb{Z})$, we will now see that a better understanding of $\mathrm{Eul}(X)$ can be obtained by observing that H acts on $\mathrm{Eul}(X)$. To define this action, suppose $h\in H_1(X)$, $e\in\mathrm{Eul}(X)$. Taking representatives, we have $h=[\widetilde{h}]$, $e=[\xi]$, where $\widetilde{h},\xi$ are respectively a 1-cycle and an Euler chain. Define $h\cdot e=[\widetilde{h}+\xi]\in\mathrm{Eul}(X)$, since $\partial(\widetilde{h}+\xi)=0+\partial\xi$. It is clear that this is well-defined, i.e. independent of the choice of $\widetilde{h}$, since any two choices must differ by a boundary.

It will henceforth be convenient to write H multiplicatively, even though it is an abelian group, since we are dealing with this action and later with the group ring of H. With this convention, it is trivial to check that $h\cdot(h_1\cdot e)=(hh_1)\cdot e$, so that this is indeed an action of H on $\mathrm{Eul}(X)$.

Proposition 2 *The above action of H on* $\mathrm{Eul}(X)$ *is transitive and free.*

Proof. If ξ,η are two Euler chains, then $\widetilde{h}=\xi-\eta$ is a 1-cycle. Clearly $[\widetilde{h}]\cdot[\eta]=[\xi]$, proving transitivity. And if $h\cdot[\xi]=[\xi]$, then with the above notations $\widetilde{h}+\xi$ and ξ differ by a boundary, i.e. $\widetilde{h}$ is itself a boundary and therefore its homology class $[\widetilde{h}]$ is trivial. Thus the action is also free. ■

Corollary 2.1 $|\mathrm{Eul}(X)|=|H|$.

The next goal is to show that the choice of base points in the open cells does not affect $\mathrm{Eul}(X)$, in the sense that there is a bijective correspondence between the Euler chains which induces a canonical bijective correspondence between the Euler structures. To obtain this correspondence, recall that each open cell e_α^n in a CW-complex has a homeomorphism (the inverse of its attaching map) with the open n-disc $\overset{o}{D}^n$. Thus, for any two selected points $\underline{e}^n_\alpha$, $\underline{\varepsilon}^n_\alpha \in e^n_\alpha$, there is an oriented path β^n_α in e^n_α from $\underline{e}^n_\alpha$ to $\underline{\varepsilon}^n_\alpha$, obtained as the image of the line segment joining the two corresponding points in $\overset{o}{D}^n$ under the attaching map. For any Euler chain ξ for the choice of base points $\underline{e}^n_\alpha$, it is clear that $\eta = \xi + \sum(-1)^n \beta^n_\alpha$ is an Euler chain for the choice of base points $\underline{\varepsilon}^n_\alpha$, and thus the correspondence mentioned above is obtained. This correspondence is obviously H-equivariant; this amounts to nothing more than the commutativity of addition.

Now let us show that for a CW-subdivision X' of X, there is again a canonical bijective correspondence $\mathrm{Eul}(X) = \mathrm{Eul}(X')$, which is also H-equivariant. In fact this is rather similar to the preceding argument (of course, since $H_1(X) = H_1(X')$, the above proposition already shows that $\mathrm{Eul}(X)$ and $\mathrm{Eul}(X')$ are in bijective correspondence as H-sets, but we would like to strengthen this to a canonical isomorphism of H-sets). For simplicity let $a = e^n_\alpha$ denote an open n-cell of X and $\underline{a}$ its base point. In X', a is subdivided into a finite number of cells which we denote $b_1, ..., b_k$. Also let the dimension of b_i be n_i (of course $n_i \leq n$), and its base point is written $\underline{b}_i$. As in the previous paragraph, for each i one chooses an oriented path β_i in a with $\partial\beta_i = \underline{a} - \underline{b}_i$. Now define the 1-chain $\zeta_a = \sum_{i=1}^k (-1)^{n_i} \beta_i$. To determine $\partial\zeta_a$, the following elementary lemma will be useful.

Lemma 4 *Let $B = B^n$ be given a CW-structure, with each open cell contained entirely in $\partial B = S^{n-1}$ or in the interior $\overset{o}{B} = B \setminus \partial B$. Then $\chi(\overset{o}{B}) = (-1)^n$.*

Proof. Since $\chi(\overset{o}{B}) + \chi(S^{n-1}) = \chi(B^n) = 1$, we find that $\chi(\overset{o}{B}) = 1 - (1 - (-1)^n) = (-1)^n$. ■

Returning now to ζ_a, and using this lemma, we have

$$\begin{aligned} \partial\zeta_a &= \textstyle\sum_{i=1}^k (-1)^{n_i} \partial\beta_i = \sum_{i=1}^k (-1)^{n_i} (\underline{a} - \underline{b}_i) \\ &= \chi(\overset{o}{B})\underline{a} - \textstyle\sum_{i=1}^k (-1)^{n_i} \underline{b}_i \\ &= (-1)^n \underline{a} - \textstyle\sum_{i=1}^k (-1)^{n_i} \underline{b}_i. \end{aligned}$$

From this calculation, it is clear that starting from an Euler chain ξ for X, taking $\xi' = \xi - \sum \zeta_a$, where the sum is taken over all open cells $a = e^n_\alpha$ of X, one obtains an Euler chain for X'. Just as in the previous argument, this correspondence is canonical and induces a canonical correspondence on the Euler structures $\mathrm{Eul}(X) = \mathrm{Eul}(X')$ that is H-equivariant.

In Lecture 3 the notion of a fundamental family of cells $\widetilde{e}^{n_i}_i$ for the maximal abelian cover $\widetilde{X}$ of X was defined. With any such fundamental family, an Euler structure for X is determined, as follows. First, let $x \in \widetilde{X}$ be any point, and next select base points $\underline{\widetilde{e}}^{n_i}_i \in \widetilde{e}^{n_i}_i$. Let $p : \widetilde{X} \twoheadrightarrow X$ be the covering projection, then take $\underline{e}^{n_i}_i = p(\underline{\widetilde{e}}^{n_i}_i)$ as the base point for $e^{n_i}_i$. Since $\widetilde{X}$ is connected, we may choose an oriented arc α_i from x to $\underline{\widetilde{e}}^{n_i}_i$, for each fundamental cell $\widetilde{e}^{n_i}_i$ of $\widetilde{X}$. Finally, define $\xi = \sum_i (-1)^{n_i} p\alpha_i$. We claim that ξ is an Euler chain for X, as follows from the simple computation:

$$\partial\xi = \sum_i (-1)^{n_i} [p(\underline{\widetilde{e}}^{n_i}_i) - p(x)] = \sum_i (-1)^{n_i} \underline{e}^{n_i}_i - \chi(X) p(x) = \sum_i (-1)^{n_i} \underline{e}^{n_i}_i,$$

recalling that $\chi(X) = 0$.

The Euler structure $[\xi] \in \mathrm{Eul}(X)$ is independent of the choices made, both for the arcs α_i and for x. To see the independence for the choice of arcs, suppose α_i' is another choice. Then $\alpha_i - \alpha_i'$ determines a 1-cycle in $\widetilde{X}$ and thus a 1-dimensional homology class in $H_1(\widetilde{X})$. But because $\widetilde{X}$ is a maximal abelian cover of X, the induced map in homology $p_* : H_1(\widetilde{X}) \to H_1(X)$ equals 0. It follows that $p(\alpha_i)$ and $p(\alpha_i')$ are homologous (i.e., differ by a boundary), and therefore give rise to the same Euler structure on X (recall that this was precisely the equivalence relation used to define an Euler structure).

Secondly, suppose a different point $y \in \widetilde{X}$ is selected. Using connectivity choose a fixed path β from y to x. To get an Euler structure $[\eta]$ starting at y, since we have already shown independence of the choice of paths, we may select the path from y to $\underset{\sim}{\widetilde{e}}{}_i^{n_i}$ that consists of the path β followed by α_i. This path is homologous to $\beta + \alpha_i$, so the Euler structure is given by

$$[\eta] = [\sum_i (-1)^{n_i}(p\beta + p\alpha_i)] = [\sum_i (-1)^{n_i} p\alpha_i] = [\xi],$$

using once more the fact that $\chi(X) = 0$.

Summarizing the above discussion, we have shown that for any fundamental family $\mathcal{F}$ of cells in $\widetilde{X}$, a unique Euler structure $e = e(\mathcal{F}) \in \mathrm{Eul}(X)$ is determined. Conversely, we have the next proposition.

Proposition 3 *For any $e \in \mathrm{Eul}(X)$, there exists a fundamental family of cells $\mathcal{F}$ in $\widetilde{X}$ such that $e = e(\mathcal{F})$.*

Proof. Start with any fundamental family of cells $\mathcal{F}_1$, which determines an Euler structure $e_1 = e(\mathcal{F}_1) = [\xi_1]$. By Proposition 2, there exists a unique $h \in H$ such that $e = h \cdot e_1$. Now consider any 0-cell $\widetilde{a} = \widetilde{e}_i^0 \in \mathcal{F}_1$, and also the 0-cell $b = h \cdot a$. There is a path γ in $\widetilde{X}$ from a to b, and $p\gamma$ is a loop in X, i.e. a 1-cycle, that represents h. It is then clear that modifying $\mathcal{F}_1$ on the single cell a to $\mathcal{F} = (\mathcal{F}_1 \setminus \{a\}) \cup \{b\}$ has the effect of adding $p\gamma$ to the Euler chain ξ_1, and thus gives the Euler structure $[p\gamma + \xi_1] = h \cdot e_1 = e$. ∎

Remark. There is no uniqueness of the fundamental family of cells $\mathcal{F}$ in Proposition 3. However, a slight modification of the proof of Proposition 3, left to the reader as an exercise, shows that if $\mathcal{F} = (e_\alpha^n)_{n,\alpha}$ and $\mathcal{F} = ({e'}_\alpha^n)_{n,\alpha}$ are two fundamental families of cells such that $e = e(\mathcal{F}) = e(\mathcal{F}')$, then

$$\prod (h_\alpha^n)^{(-1)^n} = 1,$$

where the product is taken over all cells of all dimensions and $h_\alpha^n \in H$ is the unique element h in H such that ${e'}_\alpha^n = h \cdot e_\alpha^n$. (Recall that we use the multiplicative notation for elements in H.)

With this information on Euler structures, we can now return to the idea of torsion. Let X be any finite connected CW-complex, $\widetilde{X}$ its maximal abelian cover, and ω a homology orientation for X, as defined in Lecture 3. Also take $\mathbb{F}$ to be a field, $e \in \mathrm{Eul}(X)$ an Euler structure on X, $H = H_1(X)$ as usual, and finally let $\varphi : \mathbb{Z}[H] \to \mathbb{F}$ a homomorphism (of rings with unity, i.e. $\varphi(1) = 1$). With this data we can now define a refined torsion $\tau^\varphi(X, \omega, e) \in \mathbb{F}$ as follows. First consider the chain complex $C^\varphi(X) = \mathbb{F} \otimes_{\mathbb{Z}[H]} C_*(\widetilde{X})$, where $\mathbb{Z}[H]$ acts on $C_*(\widetilde{X})$ via the deck transformations (as discussed in Lecture 3), and acts on $\mathbb{F}$ via φ. Specifically, for any $h \in H$, $\xi \in C_*(\widetilde{X})$, $\gamma \in \mathbb{F}$, one has $\gamma \otimes (h \cdot \xi) = (\varphi(h) \cdot \gamma) \otimes \xi$, and this action is extended linearly to all of $\mathbb{Z}[H]$. The algebraic machinery developed in Lecture 2 can now be applied, once a basis is chosen for our chain complex $C^\varphi(X)$. Of course this is given by the last proposition, i.e. one takes a fundamental family of cells $\mathcal{F}$ such that $e = e(\mathcal{F})$ for the basis. We can now define

$$\tau^\varphi(X, \omega, e) = \begin{cases} \tau(X, \omega, \mathcal{F}) \in \mathbb{F}^*, & \text{if } C^\varphi(X) \text{ is acyclic,} \\ 0 \in \mathbb{F}, & \text{otherwise.} \end{cases}$$

Here $\tau(X, \omega, \mathcal{F})$ denotes the sign-refined torsion defined by means of the homology orientation ω of $H_*(X; \mathbb{R})$ and the fundamental family of cells $\mathcal{F}$ in Lecture 3. If one chooses another fundamental family $\mathcal{F}'$ of cells such that $e = e(\mathcal{F})$, then

$$\tau(X, \omega, \mathcal{F}') = \varphi\left(\prod (h_{\alpha,n})^{(-1)^n}\right) \tau(X, \omega, \mathcal{F}) = \tau(X, \omega, \mathcal{F}),$$

according to the Remark following Proposition 3. (The first equality above is a general consequence of how the torsion changes under cell change, see Lecture 3, p. 18.)

This achieves the goal stated in Lecture 3, namely by fixing the homology orientation and the Euler structure, the torsion is a well-defined unique element of $\mathbb{F}$. The next proposition gives a nice equivariance property of τ^φ with respect to the action of H.

Proposition 4 *For $h \in H$, one has $\tau^\varphi(X, \omega, h \cdot e) = \varphi(h) \cdot \tau^\varphi(X, \omega, e)$.*

Proof. The proof is quite similar to that of the previous proposition. One simply modifies a fundamental family of cells for e on a single 0-cell a to $h \cdot a$. This gives a fundamental family of cells for $h \cdot e$, and it is easily checked that this multiplies the torsion by $\varphi(h)$. ■

Thus far X has represented an arbitrary finite CW-complex. We now turn to the important special case $X = M^m$, a smooth (C^∞) manifold of dimension m. It will be supposed that M is closed (compact and without boundary) and connected. We are interested in non-singular vector fields u on M, meaning that for each $x \in M$, $u(x)$ is a non-zero tangent vector at x, and furthermore the function u is continuous (even smooth). Not every such manifold admits a non-singular vector field, for example S^2 has no non-singular vector field ("you can't smooth down the hair on a hairy basketball", or equivalently "at some point on the Earth the wind is not blowing"). In fact, there is a theorem of the Swiss mathematician Heinz Hopf, from 1927 [3], that asserts that M admits a non-singular vector field if and only if $\chi(M) = 0$. For example, as we saw in Lecture 1, $\chi(S^{2k}) = 2 \neq 0$, hence any even-dimensional sphere cannot admit a non-singular vector field. Nevertheless the family of manifolds for which a non-singular vector field exists is large, indeed for any odd dimensional manifold M it follows from Poincaré duality that $\chi(M) = 0$ and therefore M admits a non-singular vector field.

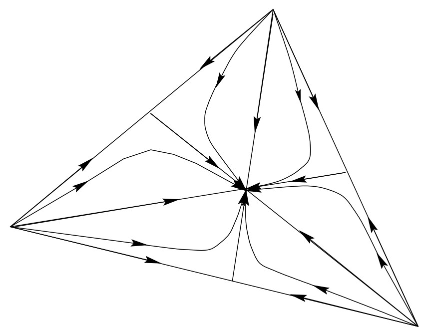


Figure 14: The Stiefel flow on a 2-simplex

Definition 4.3. $\mathrm{Vect}(M) = \mathcal{V}/\sim$, where $\mathcal{V}$ is the set of all non-singular vector fields on M, and the equivalence relation is defined by $u \sim v$ if and only if the restrictions of u and v to the complement of some point in M are homotopic through non-singular vector fields.

Note that one is led to the same equivalence relation, and therefore to the same definition of $\mathrm{Vect}(M)$, if one replaces in the definition above "the complement of some point of M" by "the complement of some ball in M" or by "the complement of any point of M".

We now state a theorem which gives another interpretation of $\mathrm{Eul}(M)$ for smooth manifolds. Here we are making use of a basic theorem of differential topology, which says that for a smooth manifold M there exists a smooth (C^1) triangulation of M, and thus M has a CW-structure with respect to which $\mathrm{Eul}(M)$ can be defined.

Theorem 12 *For a smooth manifold M with $\chi(M) = 0$, $\mathrm{Vect}(M) = \mathrm{Eul}(M)$.*

Proof outline. The proof consists in defining a map $\mathrm{Eul}(M) \to \mathrm{Vect}(M)$ and verifying that it is $H_1(M)$-equivariant. The smooth manifold M has a C^1 triangulation which we call X. To the first barycentric subdvision $X^{(1)}$ of X, one can associate the so-called Stiefel-Whitney singular vector field. This vector field is smooth, and non-singular except precisely at the barycenters of the simplices. For an explicit definition, the reader is referred to [2] and [18]. The picture of the flow on a 2-simplex is given in Fig. 14. Let $\mathcal{F} = \{\tilde{e}_i\}$ be a fundamental family of simplices in the maximal abelian covering $\tilde{X}$ of X. Each simplex $\tilde{e}_i$ in $\tilde{X}$ covers a simplex e_i in X. For each i, choose a path in $\tilde{X}$ from the base point $\tilde{x}_0 \in \tilde{X}$ to the barycenter of $\tilde{e}_i$. The projection of this path to X is a path from the base point $x_0 \in X$ to e_i. These paths can be chosen so that they are disjointly embedded except at x_0. Thus the union of these paths for all i is a wedge of embedded intervals in X, a regular neighborhood of which is a ball B. The vector field being non-singular on the complement of B, it can be extended to a non-singular vector field ν to the whole manifold M (since $\chi(M) = 0$). The class of ν in $\mathrm{Vect}(M)$ only depends on $\mathcal{F} \in \mathrm{Eul}(X) = \mathrm{Eul}(M)$. This yields the desired map $\mathrm{Eul}(M) \to \mathrm{Vect}(M)$, $\mathcal{F} \to \nu$. This map is $H_1(M)$-equivariant and hence bijective. ■

Before specializing this still further to 3-manifolds, we introduce the idea of a Spin^c-structure on M. For $n \geq 3$, it is well known that one has $\pi_1(SO(n)) = \mathbb{Z}_2$, so the universal covering of $SO(n)$ is a 2-fold covering. It is also a Lie group and is called $\mathrm{Spin}(n)$. Thus, for $n \geq 3$, one has an exact sequence of Lie groups

$$\mathbb{Z}_2 \hookrightarrow \mathrm{Spin}(n) \overset{\phi}{\twoheadrightarrow} SO(n),$$

with $\mathrm{Spin}(n)$ simply connected. The group $\mathbb{Z}_2 = \{\pm 1\}$ acts on $\mathrm{Spin}(n)$ via group multiplication by ± 1, thus $\phi(x) = \phi(-x)$, $x \in \mathrm{Spin}(n)$. We now briefly explain how the group $\mathrm{Spin}(n)$ is related to Clifford algebras. This can be found in many textbooks. A good source, for example, is [5], since this covers Clifford algebras, $\mathrm{Spin}(n)$, as well as $\mathrm{Spin}^c(n)$ (in Appendix D of this book). The perhaps easiest way to define Clifford algebras is in terms of generators and relations. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. The Clifford algebra $\mathrm{Cl}(n)$ is the algebra over $\mathbb{R}$ generated by $e_1, \ldots, e_n$ subject to relations

$$e_i \cdot e_j = -e_j \cdot e_i \text{ for } i \neq j \text{ and } e_i^2 = -1. \tag{3}$$

It follows from the definition that there is a natural inclusion $\mathbb{R}^n \subset \mathrm{Cl}(n)$.

Writing every element of $\mathrm{Cl}(n)$ as a sum of ordered products $e_{i_1} \cdots e_{i_r}$ with $i_1 < \cdots < i_r$, we see that

$$\dim_{\mathbb{R}} \mathrm{Cl}(n) = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Any $\mathbb{R}$-map $f : \mathbb{R}^n \to \mathbb{R}^n$ extends to an $\mathbb{R}$-algebra map $\tilde{f} : T(n) \to T(n)$, where

$$T(n) = \oplus_{j \geq 0}(\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n) = \oplus_{j \geq 0}(\mathbb{R}^n)^{\otimes j}$$

is the tensor algebra generated by $\mathbb{R}^n$. If f is orthogonal (i.e. preserves the euclidean norm of $\mathbb{R}^n$), it follows from (3) that $\tilde{f}$ induces a map of Clifford algebras $\mathrm{Cl}(n) \to \mathrm{Cl}(n)$. It follows that there is an embedding $O(n) \hookrightarrow \mathrm{Aut}(\mathrm{Cl}(n))$.

In particular, the map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto -x$ extends to an involution $\alpha : x \mapsto -x$ of $\mathrm{Cl}(n)$ which yields a decomposition

$$\mathrm{Cl}(n) = \mathrm{Cl}_0(n) \oplus \mathrm{Cl}_1(n), \tag{4}$$

where $\mathrm{Cl}_j(n) = \mathrm{Ker}\ (\alpha - (-1)^j \mathrm{Id})$. Let $\mathrm{Pin}(n)$ be the group generated by elements v in $\mathbb{R}^n \subset \mathrm{Cl}(n)$ such that $||v||^2 = 1$. Any element $v \in \mathbb{R}^n \subset \mathrm{Pin}(n)$ is invertible : decompose a generator $v = \sum_i \lambda_i e_i$, so according to (3),

$$v^2 = \sum_{i,j} \lambda_i \lambda_j e_i e_j = \sum_i \lambda_i^2 e_i^2 + \sum_{i \neq j} \lambda_i \lambda_j e_i e_j = -\sum_i \lambda_i^2 = -||v||^2 = -1.$$

Thus $\mathrm{Pin}(n)$ is a subgroup of the multiplicative group of units (invertible elements) of $\mathrm{Cl}(n)$. The classical spin group is defined by

$$\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cl}_0(n).$$

It consists of all elements of $\mathrm{Pin}(n)$ which can be written as a product of an even number of the generators given above for $\mathrm{Pin}(n)$. Because any product of elements of an orthonormal basis is in Pin(n), Pin(n) contains a vector space basis for Cl(n). Similarly, Spin(n) contains a vector space basis for $\mathrm{Cl}_0(n)$. It follows that (the isomorphism class) of any representation ρ of $\mathrm{Cl}_0(n)$ is determined by (the isomorphism class of) its restriction $\rho|_{\mathrm{Spin}(n)}$.

To see that this definition is equivalent to the one presented above, note that Spin(n) acts on Cl(n) by conjugation. This action preserves the algebra structure and the decomposition (4). A little more careful observation shows that this action induces a representation of Spin(n) as endomorphisms of Cl(n) preserving $\mathbb{R}^n \subset \mathrm{Cl}(n)$, acting on $\mathbb{R}^n$ as elements of SO(n). Hence there is an induced map $\mathrm{Spin}(n) \to SO(n)$, easily seen to be surjective, with kernel ± 1.

The particular case $n = 3$ will be of special interest to us (see Lecture 5). The Clifford algebra Cl(2) has dimension 4 and is easily seen from (3) to be isomorphic to the quaternion algebra $\mathbb{H}$. The Clifford algebra Cl(3) has dimension 8 and is isomorphic to the direct sum of two copies of the quaternion algebra $\mathbb{H}$. The subalgebra $\mathrm{Cl}_0(3)$ is isomorphic to Cl(2), thus to $\mathbb{H}$ (in fact diagonally embedded in $\mathrm{Cl}(3) = \mathbb{H} \oplus \mathbb{H}$). The group Spin(3) is the group of unit quaternions, isomorphic to S^3.

We now define $\mathrm{Spin}^c(n)$, the "complexification" of Spin(n). Let S^1 be the unit circle, thought of as the set of complex numbers of modulus 1. Then

$$\mathrm{Spin}^c(n) = (\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts via the diagonal embedding, specifically $[x, \lambda] = [-x, -\lambda]$, where $x \in \mathrm{Spin}(n)$, $|\lambda| = 1$. It is then clear that there is an exact sequence of Lie groups

$$S^1 \overset{i}{\hookrightarrow} \mathrm{Spin}^c(n) \overset{\psi}{\twoheadrightarrow} SO(n) \tag{5}$$

where $i(\lambda) = [1, \lambda]$ and $\psi[x, \lambda] = \phi(x)$ (here $1 \in \mathrm{Spin}(n)$ is taken as the base point of Spin(n)).

Exercise. Any complex representation ρ of Spin(n) such that $\rho(-1) = -1$ extends uniquely to a complex representation of $\mathrm{Spin}^c(n)$.

We record a special example in dimension 3.

Example. $\mathrm{Spin}^c(3) \approx U(2)$ as Lie groups.

To see this, first recall the familiar isomorphisms $SU(2) \approx S^3 \approx \mathrm{Spin}(3) \approx Sp(1)$. Assuming this, we think of an element x of Spin(3) as a matrix in $SU(2)$, i.e.

$$x = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},$$

where $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$. Representing an element of S^1 as usual by a unit complex number λ, it is not hard to check that the map

$$[x, \lambda] \mapsto \lambda \cdot x = \begin{bmatrix} \lambda a & \lambda b \\ -\lambda\bar{b} & \lambda\bar{a} \end{bmatrix}$$

is well defined, is a homomorphism, and is bijective, as a map from $\mathrm{Spin}^c(3) \to U(2)$. ■

The complex Clifford algebra is defined as $\mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathbb{C}$. Denote by $\mathfrak{M}(\mathbb{C}, k)$ the algebra over $\mathbb{C}$ of $k \times k$ matrices of complex numbers. A basic result asserts that

$$\mathrm{Cl}(n) \otimes \mathbb{C} \approx \begin{cases} \mathfrak{M}(\mathbb{C}, 2^{\frac{n}{2}}) & \text{if } n \text{ is even;} \\ \mathfrak{M}(\mathbb{C}, 2^{\frac{n-1}{2}}) \oplus \mathfrak{M}(\mathbb{C}, 2^{\frac{n-1}{2}}) & \text{if } n \text{ is odd.} \end{cases} \tag{6}$$

Furthermore, if n is odd then $\mathrm{Cl}_0(n) \otimes \mathbb{C}$ is embedded diagonally in the sum (6).

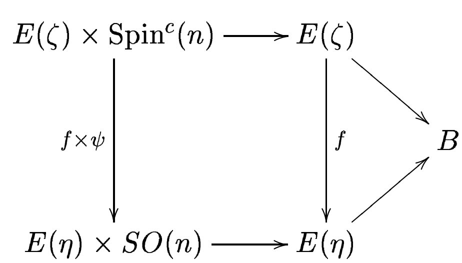
By Wederburn's theorem, $\mathfrak{M}(\mathbb{C}, k)$ has an irreducible finite dimensional complex representation $S_{\mathbb{C}}$, unique up to isomorphism and the corresponding map $\mathfrak{M}(\mathbb{C}, k) \to \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}})$ is an isomorphism of algebras. It follows from (6) that

- If n is even, then $\mathrm{Cl}(n)$ has an irreducible complex representation $S_{\mathbb{C}}(n)$ of dimension $2^{\frac{n}{2}}$, which is unique up to isomorphism. In this case, the action of $\mathrm{Cl}(n) \otimes \mathbb{C}$ on $S_{\mathbb{C}}(n)$ induces an isomorphism
$$\mathrm{Cl}(n) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}}(n)) = S_{\mathbb{C}}(n) \otimes S_{\mathbb{C}}(n)^*. \tag{7}$$
- If n is odd, then $\mathrm{Cl}(n)$ has exactly two irreducible complex representations, up to isomorphism (obtained by projecting $\mathrm{Cl}(n) \otimes \mathbb{C}$ onto one of the two summands in (6) and taking the Wederburn's irreducible representation of that summand), each of them of dimension $2^{\frac{n-1}{2}}$. These representations induce, by restriction, isomorphic irreducible representations $S_{\mathbb{C}}(n)$ of $\mathrm{Cl}_0(n)$. The action of $\mathrm{Cl}_0(n) \otimes \mathbb{C}$ on $S_{\mathbb{C}}(n)$ induces an isomorphism
$$\mathrm{Cl}_0(n) \otimes \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}}(n)) = S_{\mathbb{C}}(n) \otimes S_{\mathbb{C}}(n)^*. \tag{8}$$

Note that by the exercise above, the induced representation $\mathrm{Spin}(n) \to \mathrm{Aut}_{\mathbb{C}}(S_{\mathbb{C}}(n))$ extends uniquely to a representation $\mathrm{Spin}^c(n) \to \mathrm{Aut}_{\mathbb{C}}(S_{\mathbb{C}}(n))$

Now consider the tangent bundle $\tau_M \to M$ for a smooth n-dimensional manifold M. In general the structure group $GL(n)$ of this bundle can be reduced to $O(n)$, e.g. by the introduction of a Riemannian metric. If the manifold is orientable it can be further reduced to $SO(n)$ (this is equivalent to the vanishing of the first Stiefel-Whitney class $w_1(M)$), and if the structure group lifts further to $\mathrm{Spin}(n)$ (equivalent to the vanishing of $w_2(M)$) one then calls the orientable manifold a spin manifold. Similarly it is also possible that the structure group may lift to $\mathrm{Spin}^c(n)$, and one then calls M a Spin^c manifold.

Definition 4.4. Let $\eta : E(\eta) \to B$ be a principal right $SO(n)$-bundle over base B. A *rigid* Spin^c*-structure* on η is a pair (ζ, f), where $\zeta : E(\zeta) \to B$ is a principal $\mathrm{Spin}^c(n)$-bundle over B and $f : E(\zeta) \to E(\eta)$ is a map such that the following diagram is commutative



where $\psi : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ is the canonical epimorphism defined by (5) and the horizontal arrows are the right actions of $\mathrm{Spin}^c(n)$ and $\mathrm{SO}(n)$ respectively. Two such pairs (ζ, f) and (ζ', f') define the *same* Spin^c-*structure* if there exists a bundle isomorphism $g : \zeta \to \zeta'$ such that $f' \circ g = f$. A Spin^c-*structure* on an oriented n-manifold is a Spin^c-structure on the oriented frame bundle F_M associated to T_M.

The obstruction to the existence of Spin^c-structures on M is $\beta w_2(M) \in H^3(M)$ where $\beta : H^2(M; \mathbb{Z}_2) \to H^2(M)$ is the Bockstein homomorphism associated to the short exact sequence of coefficients $0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ and $w_2(M) \in H^2(M; \mathbb{Z}_2)$ is the second Stiefel-Whitney class of M. By exactness of the sequence

$$\cdots \to H^2(M) \to H^2(M; \mathbb{Z}_2) \stackrel{\beta}{\to} H^3(M) \to \cdots,$$

we see that Spin^c-structures exist on M if and only if $w_2(M)$ is the mod 2 reduction of an integral cohomology class (in $H^2(M) = H^2(M; \mathbb{Z})$). Hence it is "easier" for an orientable manifold to be Spin^c than to be spin. For further details see Appendix D of [5].

Turning to the case $M = M^3$ of an orientable 3-manifold, a theorem of Stiefel [15] implies that τ_M is in fact trivial – one also says that M is parallelizable or equivalently that M admits 3 pointwise linearly independent vector fields. One consequence of this is that M is a Spin^c manifold. Another useful fact is that $H^2(M)$ acts freely and transitively on the set of Spin^c structures on M; this is related to the fact that the first Chern class $c_1(M)$ classifies circle bundles over M.

Theorem 13 *For M an oriented 3-manifold,* $\mathrm{Spin}^c(M) = \mathrm{Vect}(M)$.

Proof outline. Let v be a non-singular vector field on M, which certainly exists since the tangent bundle (as mentioned above) is trivial. Also suppose that M has been supplied with a Riemannian metric. Then using v one obtains an orthogonal decomposition of the tangent bundle $\tau_M = \mathbb{R}v \oplus (\mathbb{R}v)^{\perp}$. The vector bundle $\mathbb{R}v$ is of course a trivial line bundle, with trivial structure group. The complementary vector bundle $(\mathbb{R}v)^{\perp}$ is of rank 2, and its structure group is $S^1 = SO(2) \hookrightarrow U(2) = \mathrm{Spin}^c(3)$. ■

Corollary 13.1 *For M as above,* $\mathrm{Spin}^c(M) = \mathrm{Eul}(M)$.

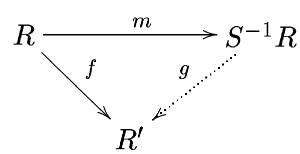
We end Lecture 4 at this point.

Lecture 5. Relations with the Seiberg-Witten Invariants

The aim of this lecture is to define a *numerical* invariant of Spin^c-structures (or, equivalently, Euler structures), following [19]. We shall also discuss relations with the Seiberg-Witten Invariants.

The first step consists in defining the "maximal abelian torsion". We need a few classical algebraic preliminaries.

Definition 5.1. Let R be a commutative ring (with unit 1) and S a subset of R. The localization $S^{-1}R$ is a commutative R-algebra and an R-algebra map $m : R \to S^{-1}R$ such that $m(s)$ is invertible in $S^{-1}R$ for every $s \in S$ and $(S^{-1}R, m)$ solves the universal problem:



i.e., for any commutative R-algebra R' and any R-algebra map $f : R \to R'$ such that $f(s)$ is invertible for any $s \in S$, there exists a unique R-algebra map $g : S^{-1}R \to R$ such that $g \circ m = f$.

As is well-known, the localization $S^{-1}R$ exists and is unique up to isomorphism, for any subset $S \subseteq R$. If S is multiplicatively closed and contains $1 \in R$, then any element $x \in S^{-1}R$ can be written (not uniquely in general) $x = m(r)m(s)^{-1}$ for some $r \in R$ and $s \in S$. The kernel Ker m consists precisely of elements of R that are annihilated by multiplication by elements in S.

Definition 5.2. Let R be a commutative ring (with unit 1). Denote by S the subset of all non-zero divisors of R. The localization $S^{-1}R$ is called the *classical ring of fractions* of R and is denoted $Q(R)$.

It follows that the ring homomorphism $m : R \to Q(R)$ is injective (so R may be regarded as a subring of $Q(R)$) and that if R is a domain (i.e. has no non-zero divisors), then $Q(R)$ coincides with the quotient field $\mathrm{Fr}(R)$ of R.

Let X be a finite connected CW-complex. Finiteness implies that the abelian group $H = H_1(X)$ is finitely generated. The group ring $\mathbb{Z}[H]$ is an associative commutative ring with unit.

Suppose first that H is free abelian, generated by $t_1, \dots, t_n$. (We use multiplicative notation.) Then $\mathbb{Z}[H]$ is isomorphic to the ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials in n variables. Since $\mathbb{Z}$ is a domain, $\mathbb{Z}[t_1, \dots, t_n]$ is also a (unique factorization) domain. It follows that $\mathbb{Z}[H]$ is again a (unique factorization) domain. In particular, $Q(\mathbb{Z}[H]) = \mathrm{Fr}(\mathbb{Z}[H])$ which is isomorphic to $\mathbb{Q}(t_1, \dots, t_n)$, the field of rational functions of $t_1, \dots, t_n$ with coefficients in $\mathbb{Q}$.

In general, if H has torsion elements, then $\mathbb{Z}[H]$ has zero divisors and therefore so has $Q(\mathbb{Z}[H])$. In particular, we cannot apply the previous argument. Nevertheless, the following result holds.

Lemma 5 *Let H be a finite abelian group. Then $\mathbb{Q}[H]$ splits in a unique fashion as a direct sum of finitely many fields.*

The fields in question are cyclotomic: they appear naturally as one extends characters of H to ring homomorphisms $\mathbb{Q}[H] \to \mathbb{C}$. Since $Q(\mathbb{Z}[H]) = Q(\mathbb{Q}[H])$, lemma 5 easily implies that $Q(\mathbb{Z}[H]) = \mathbb{Q}[H]$ for any finite abelian group.

To treat the general case, choose a splitting $H = \text{Tors } H \oplus G$ where $G = H/\text{Tors } H$. By lemma 5, $\mathbb{Q}[\text{Tors } H] = \oplus_{j=1}^{r} C_j$ where C_j, $j = 1, \ldots, r$, is a field. Thus

$$\mathbb{Q}[H] = (\mathbb{Q}[\text{Tors } H])[G] = \oplus_{j=1}^{r} C_j[G] = \oplus_{j=1}^{r} R_j$$

where $R_j = C_j[G]$, $j = 1, \ldots, r$, is a domain (since G is free abelian and C_j is a domain). Therefore $\mathbb{Q}[H]$ is a direct sum of finitely many domains.

Remark. The splitting $H = \text{Tors } H \oplus G$ is not unique. However, the splitting of $\mathbb{Q}[H]$ obtained above is unique.

As a consequence, $Q(\mathbb{Z}[H]) = Q(\mathbb{Q}[H]) = Q(\oplus_j R_j) = \oplus_j Q(R_j)$ is a direct sum of fields. Set $F_j = Q(R_j)$, $j = 1, \ldots, r$. Consider the ring inclusion $\varphi : \mathbb{Z}[H] \to Q(\mathbb{Z}[H]) = \oplus_{j=1}^{r} F_j$. We can define $\varphi_j : \mathbb{Z}[H] \to F_j$ to be φ composed with the projection onto the j-th summand. This in turns yields a torsion $\tau^{\varphi_j}(X, \omega, e) \in F_j$ for any $e \in \text{Eul}(X)$ provided that X is endowed with a homology orientation ω.

Definition 5.3. The *maximal abelian torsion* $\tau(X, \omega, e)$ is defined as

$$\tau(X, \omega, e) = \sum_{j=1}^{r} \tau^{\varphi_j}(X, e) \in \bigoplus_{j=1}^{r} F_j. \tag{9}$$

As we know from the previous two lectures, $\tau(X, \omega, e)$ has no indeterminacy at all and is well defined as an element in $Q(\mathbb{Z}[H]) = \bigoplus_{j=1}^{r} F_j$. To simplify notation, we shall omit the homology orientation and write $\tau(M, e) = \tau(M, \omega, e)$. Recall also from the previous lecture that H acts on $\text{Eul}(X)$. Then for $h \in H \subset Q(\mathbb{Z}[H])$, Proposition 4 now reads

$$\tau(X, h \cdot e) = h\tau(X, e). \tag{10}$$

We now specialize further to the case when X is a closed oriented 3-manifold M. A priori, as we have seen, $\tau(M, e)$ lies in $Q(\mathbb{Z}[H])$. But since M is a 3-manifold, there is a stronger result.

Theorem 14 *Suppose that $b_1(M) = \text{rank } H \geq 2$. For any $e \in \text{Eul}(X)$, $\tau(M, e) \in \mathbb{Z}[H]$.*

In particular, we can expand $\tau(M, e) = \sum_{g \in H} n_g\, g$ where $n_g \in \mathbb{Z}$. Each integer $n_g = n_g(e)$ depends on the Euler structure e.

Definition 5.4. The *torsion function* $T : \mathrm{Eul}(M) \to \mathbb{Z}$ is defined by $T(e) = n_1(e)$. (Recall that as H is regarded as a multiplicative group, 1 is the neutral element of H.)

The torsion function introduced in [19] is the desired numerical invariant, as it takes values in $\mathbb{Z}$. Note that there is no loss of information with respect to the (sign-refined) maximal abelian torsion : formula (10) implies that $T(h \cdot e) = n_{h^{-1}}(e)$ and therefore

$$\tau(M, e) = \sum_{g \in G} T(g^{-1}e)\ g.$$

We now present the relation with Seiberg-Witten invariants. To formulate the Seiberg-Witten invariants for a closed 3-manifold M, we endow M with

- a Riemannian metric g,
- a Spin^c-structure σ.

Recall from the previous Lecture, that $\mathrm{Spin}(3) = U(2)$. Thus a rigid representative of σ is a principal $U(2)$-bundle $P \to M$ lifting the bundle F_M of orthonormal frames over M. There is a natural action ρ of the group $U(2)$ on $\mathbb{C}^2$: thus we can form a 2-dimensional complex vector bundle (called the *spin bundle*) by setting $S_{\mathbb{C}}(P) = P \times_\rho \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$. Note that ρ is nothing else than the Clifford representation discussed in the previous Lecture for $n = 3$ ($S_{\mathbb{C}}(3) = \mathbb{C}^2$).

From the $SO(3)$-bundle F_M of orthonormal frames over M, one can also form

$$\mathrm{Cl}(F_M) = F_M \times_{SO(3)} \mathrm{Cl}(3)$$

since $SO(3)$ acts naturally on $\mathrm{Cl}(3)$. This is a locally trivial bundle of real Clifford algebras and is defined without the choice of a spin structure. It can be regarded as a new algebra structure on the exterior algebra of the tangent bundle. Similarly, there is the *bundle of complex Clifford algebras*

$$\mathrm{Cl}(F_M) \otimes \mathbb{C} = F_M \times_{SO(3)} (\mathrm{Cl}(3) \otimes \mathbb{C}).$$

Since there is a Spin^c-structure σ (represented by the $U(2)$-bundle $P \to M$) the complex Clifford bundle $\mathrm{Cl}(F_M) \otimes \mathbb{C}$ will act on the complex spin bundle $S_{\mathbb{C}}(P)$, as we now proceed to explain. Recall that $\mathrm{Spin}^c(3) = U(2)$ acts on $\mathrm{Cl}(3)$ by conjugation. It follows that we can also regard $\mathrm{Cl}(F_M) \otimes \mathbb{C}$ as

$$\mathrm{Cl}(F_M) \otimes \mathbb{C} = P \times_{\mathrm{Spin}^c(3)} (\mathrm{Cl}(3) \otimes \mathbb{C}).$$

The Clifford multiplication of $\mathrm{Cl}(3) \otimes \mathbb{C}$ on $S_{\mathbb{C}}(3) = \mathbb{C}^2$ commutes with the action of $\mathrm{Spin}^c(3) = U(2)$ in the sense that $(\alpha \cdot c) \cdot v = (\alpha c \alpha^{-1}) \cdot v = \alpha \cdot (c \cdot v)$ for $\alpha \in \mathrm{Spin}^c(3)$, $c \in \mathrm{Cl}(3)$ and $v \in \mathbb{C}^2$. This remark leads to the observation that there is a well-defined global action

$$(\mathrm{Cl}(F_M) \otimes \mathbb{C}) \times S_{\mathbb{C}}(P) \to S_{\mathbb{C}}(P)$$

where each fibre is isomorphic to the Clifford multiplication.

Since M has a Riemannian metric g, there is a canonical identification of the tangent bundle and the cotangent bundle. Hence the action of $\mathrm{Cl}(F_M) \otimes \mathbb{C}$ on $S_{\mathbb{C}}(P)$ by Clifford multiplication determines an action of complex-valued differential forms on sections of the spin bundles.

There is also a natural action of $U(2)$ on $\mathbb{C}$ via the determinant : this yields a $U(1)$-bundle, which is the *determinant line bundle* of P, denoted by $\det\sigma$.

The Seiberg-Witten equations involve a $U(1)$-connection A on $\det\sigma$ (and its curvature 2-form F_A) and a section ψ of the spin bundle $S_{\mathbb{C}}(P)$. A standard reference for connections and their curvature in differential geometry is [4]. Since a connection geometrically is a horizontal distribution in the total space, any $SO(3)$-connection on $F_M \to M$ lifts to a $\mathrm{Spin}(3)$-connection. Since $\mathrm{Spin}^c(3) \to SO(3)$ is not a finite covering, there is no canonical way of lifting an $SO(3)$-connection on F_M to a $\mathrm{Spin}^c(3)$-connection on P. However, there is a unique connection $\tilde{\nabla}$ on P induced by the Levi-Cività connection on F_M and the connection A on $\det\sigma$.

The *Dirac operator* is a map on sections of $S_{\mathbb{C}}(P)$,

$$\partial\!\!\!/_A : C^\infty(S_{\mathbb{C}}(P)) \to C^\infty(S_{\mathbb{C}}(P))$$

defined by

$$\partial\!\!\!/_A(\psi)(x) = \sum_{i=1}^{3} e_i \cdot \tilde{\nabla}_{e_i}(\psi)(x)$$

where e_1, e_2, e_3 is an oriented orthonormal frame for the tangent space T_xM at x and dot denotes the Clifford multiplication. Orthonormality of the frames ensure that the Dirac operator is well-defined. The Dirac operator is an elliptic first order linear differential operator and its properties have been extensively studied, see e.g. [5].

The equations are:

$$(SW)\left\{\begin{array}{rcl} F_A & = & \psi \otimes \psi^* - \frac{1}{2}||\psi||^2 \mathrm{Id}. \\ \partial\!\!\!/_A(\psi) & = & 0. \end{array}\right. \tag{11}$$

The spin bundle $S_{\mathbb{C}}(P)$ has a hermitian metric, so that $S_{\mathbb{C}}(P)$ is sent to its dual via an anti-complex isomorphism. Here ψ^* denotes the hermitian dual to ψ, that is, the image of ψ under this isomorphism in the dual bundle $S_{\mathbb{C}}(P)^*$.

Denote by $C(P)$ the set of solutions of (SW). The group $G = C^\infty(M, U(1))$ acts on $C(P)$. Then, leaving aside technical details, one produces the moduli space

$$\mathfrak{M} = C(P)/G.$$

One major result (e.g. [6]) is that if $b_1(M) \geq 2$, that $\mathfrak{M}$ consists of a finite number of oriented points and is essentially independent of the metric g (the main difficulty), which shows that $\mathfrak{M}$ is a topological invariant of M. Define $SW(\sigma) \in \mathbb{Z}$ to be the algebraic sum of these points.

Theorem 15 *If* $b_1(M) \geq 2$ *then* $T = \pm SW$.

Here we identify $\mathrm{Eul}(M)$ and $\mathrm{Spin}^c(M)$ as explained in Lecture 4. The proof consists in establishing a set of properties ("axioms") defining the torsion function up to sign and showing that the Seiberg-Witten function satisfies these properties. See [20] for details.

Remark. In the case when the first Betti number of M is less or equal to 1, one can still extract topological invariants of M from a family of perturbations of the Seiberg-Witten equations. The process is then more delicate and the invariants obtained are related to the Casson-Walker invariant. See [6] [7] [9] [11]. Then the torsion can also be interpreted and generalized in the context of Floer homology, see [8] [12] [13].

Lecture 6. Relations with cohomology

In this last lecture, we turn to the relations between the torsion and cohomology. Let M be a closed connected oriented 3-manifold. In this section, homology and cohomology groups will be with integral coefficients, unless explicitly stated otherwise. The cohomology groups $H^0(M)$ and $H^3(M)$ are isomorphic to $\mathbb{Z}$. The first and second cohomology groups $H^1(M)$ and $H^2(M)$ are related by $H^1(M) = \mathrm{Hom}(H^2(M), \mathbb{Z})$. It follows from Poincaré duality that $H^1(M)$ is a free Abelian group of rank $b_1(M) = \mathrm{rank}\ H_1(M)$, the first Betti number of M. Furthermore, $H^1(M)$ carries a skew-symmetric trilinear form $f_M : H^1(M) \times H^1(M) \times H^1(M) \to \mathbb{Z}$ defined, in terms of the cup product, by

$$f_M(x, y, z) = (x \cup y \cup z)([M]) \in \mathbb{Z} \tag{12}$$

where $[M] \in H_3(M)$ denotes the fundamental class of M (the distinguished generator of $H_3(M)$).

Theorem 16 (D. Sullivan) *Any skew symmetric trilinear form on a lattice of finite rank can be realized as f_M for some oriented closed 3-manifold M.*

The proof involves surgery performed on handlebodies. See [16] for details.

We now discuss relations with torsions. From now on, we suppose that M is a closed connected oriented 3-manifold with $n = b_1(M) \geq 3$, equipped with a homology orientation ω and an Euler structure $e \in \mathrm{Eul}(M)$. We set $H = H_1(M) = H^2(M)$. Then the refined torsion $\tau(M, \omega, e)$, as defined in Lecture 4 and interpreted in Lecture 5, is an element in $\mathbb{Z}[H]$.

The map

$$\varepsilon : \mathbb{Z}[H] \to \mathbb{Z}, \ \sum_{h \in H} m_h h \mapsto \sum_{h \in H} m_h$$

is a ring homomorphism, called the *augmentation* map. The kernel $\mathcal{I} = \mathrm{Ker}\ \varepsilon$ is called the *augmentation ideal* of $\mathbb{Z}[H]$. For instance, $h - 1 \in \mathcal{I}$ for any $h \in H$. As in any group ring, there is a decreasing filtration

$$\mathbb{Z}[H] = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq \ldots \supseteq \mathcal{I}^n \supseteq \ldots.$$

The following result describes where $\tau(M, \omega, e)$ lies in this filtration depending on the first Betti number $n = \dim H_1(M; \mathbb{R})$ of M.

Theorem 17 *Consider the following two cases.*

Case 1. *Let $n \geq 4$ be even. Then $\tau(M, \omega, e) \in \mathcal{I}^{n-2}$.*

Case 2. *Let $n \geq 3$ be odd. Then $\tau(M, \omega, e) \in \mathcal{I}^{n-3}$. Furthermore, $\tau(M, \omega, e) \pmod{\mathcal{I}^{n-2}} \in \mathcal{I}^{n-3}/\mathcal{I}^{n-2}$ is determined by the skew symmetric trilinear form f_M.*

The skew symmetric trilinear form is a (homotopical) invariant of M and depends neither on ω nor on e. So we deduce the following consequence.

Corollary 17.1 *If $n \geq 3$ is odd, then $\tau(M,\omega,e) \pmod{\mathcal{I}^{n-2}}$ is a homotopical invariant of M.*

We shall describe an explicit formula relating the skew symmetric trilinear form f_M to the torsion $\tau(M,\omega,e)$. For this, we need some preliminary algebra for trilinear forms.

Let N be a free module of finite rank over a commutative associative ring R. Let $f : N \times N \times N \to R$ be a trilinear form. The (bilinear) map

$$\tilde{f} : N \times N \to N^*, \ \ \tilde{f}(x,y)(z) = f(x,y,z),$$

is the *right adjoint* map associated to f. If $(z_i)_i$ is a basis in N and $(z_i^*)_i$ is the dual basis in N^* then

$$\tilde{f}(a,b) = \sum_i f(a,b,z_i) z_i^*, \quad a,b \in N. \tag{13}$$

The expression (13) is independent of the choice of bases $(z_i)_i, (z_i^*)_i$. We can extend the adjoint map by embedding N^* into the symmetric algebra $\mathcal{S}(N^*)$ of N^*. We may regard $\mathcal{S}(N^*)$ as $R[(z_i^*)_i]$, the graded polynomial algebra on the elements of the dual basis. The grading is given by the degree. For any ordered basis $z = (z_i)_i$ of N, there is a well defined determinant

$$\det(\tilde{f}(z_i, z_j))_{i,j} \in \mathcal{S}(N^*).$$

A trilinear form $f : N \times N \times N \to R$ is said to be *alternate* if $f(x,x,y) = f(x,y,y) = f(x,y,x) = 0$ for all $x,y \in N$.

Lemma 6 *An alternate trilinear form is skew-symmetric. The converse holds if the characteristic of R is different from* 2.

Proof. The first statement follows from the observation that

$$0 = f(x+y, x+y, z) = f(x,y,z) + f(y,x,z)$$

and cyclic permutation of the variables. Conversely, let f be skew-symmetric. Then $f(x,x,z) = -f(x,x,z)$ so $2f(x,x,z) = 0$ and $f(x,x,z) = 0$. The conclusion follows by cyclic permutation of the variables. ■

Lemma 7 *If f is alternate then* $\det(\tilde{f}(z_i,z_j))_{i,j} = 0$

Proof. The matrix $A = (\tilde{f}(z_i,z_j))_{i,j}$ is alternate, i.e., satisfies $A^T = -A$ and has diagonal elements equal to 0. The size of A is equal to the rank n of N. Suppose first that n is odd. Then $\det A = \det A^T = -\det A$. If multiplication by 2 in R is invertible, then it follows that $\det A = 0$. In general, let T with $T_{ii} = 0$ for $i = 1, \ldots, n$ and $T_{ij} = -T_{ij}$ for $i > j$. Clearly $\det T \in \mathbb{Z}[T_{ij}]_{1 \leq i < j \leq n}$. Since $\mathbb{Z}[T_{ij}]_{1 \leq i < j \leq n}$ has characteristic different from 2, $\det T = 0$. Furthermore, A is a specialization of T. So $\det A$ is obtained from $\det T$ via the homomorphism $\mathbb{Z}[T_{ij}]_{1\le i<j\le n} \to R[T_{ij}]_{1\le i<j\le n}$ (induced from the unique homomorphism $\mathbb{Z} \to R$) and specialization. Hence $\det A = 0$. Suppose now that n is even. We recall the notion of Pfaffian of an alternate matrix of size $n = 2m$. Let ξ be a partition of $E = \{1, \ldots, 2m\}$ into subsets of 2 elements: $\xi : E = \coprod_{1\le l\le m} X_l$ with $X_l = \{i_l, j_l\}$. Define a permutation τ of the set E by $\tau(2l-1) = i_l$ and $\tau(2l) = j_l$, for $1 \le l \le m$. Clearly τ depends on the order $(1 \le l \le m)$ used to write the collection of subsets of the partition and the order (i_l, j_l) used to write the elements of each subset of the partition. However, the element $\mathrm{Pf}_\xi(A) = \varepsilon(\tau) \cdot \prod_{l=1}^m A_{i_l,j_l}$, where $\varepsilon(\tau)$ denotes the signature of τ, is independent of those choices. The Pfaffian of A, denoted $\mathrm{Pf}(A)$, is defined as $\mathrm{Pf}(A) = \sum_\xi \mathrm{Pf}_\xi(A)$ where ξ runs over all partitions of E into subsets of 2 elements. It is a classical result that $\det A = (\mathrm{Pf}(A))^2$. Hence it suffices to show that $\mathrm{Pf}(A) = 0$. Here $A_{ij} = \tilde{f}(z_i, z_j) = \sum_k f(z_i, z_j, z_k) z_k^*$. For a fixed partition ξ and a permutation τ representing it,

$$\mathrm{Pf}_\xi(A) = \sum_{k=(k_1,\ldots,k_m)} \varepsilon(\tau) \cdot \prod_{l=1}^m f(z_{i_l}, z_{j_l}, z_{k_l}) z_{k_1}^* \cdots z_{k_m}^*$$

where $k_1, \ldots, k_m$ run over the set E. Since ξ is a partition of E, $k_1 \in X_p = \{i_p, j_p\}$ for some $1 \le p \le m$. Since $\mathrm{Pf}_\xi(A)$ is independent of the order in which the partition $\xi : E = X_1 \coprod \cdots \coprod X_m$ is written, we can choose an ordering of the subsets $X_i's$ so that that $p = 1$. Then $f(z_{i_1}, z_{j_1}, z_{k_1}) = 0$ and $\mathrm{Pf}_\xi(A) = 0$. It follows that $\mathrm{Pf}(A) = 0$, which finishes the proof. ■

From now on, we assume that f is an alternate form. It turns out that a refined determinant can be defined for f.

Lemma 8 *Denote by z an ordered basis of N. Denote by $(\tilde{f}(z_j, z_k))_{j,k\ne i}$ the matrix obtained from $(\tilde{f}(z_j, z_k))_{j,k}$ by removing the i-th row and the i-th column.*

1. *There exists a unique element $d(z) \in \mathcal{S}^{n-3}(N^*)$ such that for any i,*

$$\det\ (\tilde{f}(z_j, z_k))_{j,k\ne i} = d(z) \cdot (z_i^*)^2 \in \mathcal{S}(N^*).$$

2. *If z, z' are two bases of N then $d(z) = [z/z']^2 \cdot d(z') \in \mathcal{S}^{n-3}(N^*)$.*

The second statement says that $d(z)$ and $d(z')$ differ by a multiplicative constant which is the square of an invertible element in R. Let us consider the special case $R = \mathbb{Z}$. Then the subgroup $\mathbb{Z}^*$ of invertible elements in $\mathbb{Z}$ is just $\{\pm 1\}$. Hence $d(z) = d(z')$ for any bases z, z' of N. This justifies the following definition.

Definition 6.1. The *determinant* $\mathrm{Det}\ f$ of an alternate trilinear form f on a free abelian group N is $\mathrm{Det}\ f = d(z)$ for some basis z of N.

Let us return now to the case when $N = H^1(M)$ and $f = f_M$ is given by the triple cup product as defined by (12). It will be convenient to remark that $N^* = \mathrm{Hom}(H^1(M), \mathbb{Z}) = H/\mathrm{Tors}\ H$. For any $h_i \in N^* = H/\text{Tors } H$, denote by $\tilde{h}_i \in H$ a lift of h_i. Since the first Betti number n is the rank of $H^1(M)$, Det $f \in S^{n-3}(N^*)$.

Theorem 18 *Suppose that* $n = b_1(M) \geq 3$ *is odd. Expand* Det $f_M = \sum_{h \in (N^*)^{n-3}} a_h h_1 \ldots h_{n-3}$, *where* $a_h \in \mathbb{Z}$. *Then*

$$\tau(M,e) = |\text{Tors } H| \cdot \sum_{h \in (N^*)^{n-3}} a_h (\tilde{h}_1 - 1) \ldots (\tilde{h}_{n-3} - 1) \mod \mathcal{I}^{n-2}. \tag{14}$$

Since $\tilde{h}_i - 1 \in \mathcal{I}$, the right hand side of (14) clearly is in $\mathcal{I}^{n-3}$. Modulo $\mathcal{I}^{n-2}$, it does not depend on the choice of lifts $\tilde{h}_j$.

Let $r \geq 2$ be a natural number. The ring structure of $\mathbb{Z}_r$ allows us to define a trilinear pairing

$$f_r : H^1(M;\mathbb{Z}_r) \times H^1(M;\mathbb{Z}_r) \times H^1(M;\mathbb{Z}_r) \to \mathbb{Z}_r, \ (x,y,z) \mapsto (x \cup y \cup z)([M]).$$

For any $r \geq 2$, f_r is skew-symmetric (because of the skew-symmetry of the cup product). If $r \geq 3$ is odd, then f_r is alternate. Otherwise, f_r may not be alternate. This can even be made a little more precise. Let r be even. Denote by $\beta_r : H^1(M;\mathbb{Z}_r) \to H^2(M;\mathbb{Z})$ the Bockstein homomorphism associated to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_r \to 0$. Let $x, y \in H^1(M;\mathbb{Z}_r)$. It can be seen that $f_r(x,x,y) = \frac{r}{2}(\beta_r(x) \cup y)([M])$.

Consider the projection map $p_r : \mathbb{Z}[H] \to \mathbb{Z}_r[H]$ obtained by reducing mod r the coefficients. We may investigate the nature of $p_r\tau = p_r(\tau(M,e))$ and ask whether an analog of Theorem 17 holds for $p_r\tau$. Consider the augmentation map $\varepsilon_r : \mathbb{Z}_r[H] \to \mathbb{Z}_r$, $\sum_{h \in H} m_h h \mapsto \sum_{h \in H} m_h$. Define $\mathcal{I}_r = \text{Ker } \varepsilon_r$. Recall that there is a decreasing filtration

$$\mathbb{Z}_r[H] = \mathcal{I}_r^0 \supseteq \mathcal{I}_r \supseteq \mathcal{I}_r^2 \supseteq \ldots \supseteq \mathcal{I}_r^n \supseteq \ldots.$$

The group $H_1(M;\mathbb{Z}_r)$ is a finitely generated $\mathbb{Z}_r$-module. Let b be the maximal number of cyclic summands $\mathbb{Z}_r$ in a direct sum decomposition of $H_1(M;\mathbb{Z}_r)$. The homology group H has a rank as a $\mathbb{Z}$-module (the first Betti number $b_1(M)$ of M). Since $H_1(M;\mathbb{Z}_r) = H \otimes \mathbb{Z}_r$, it follows that $b \geq b_1(M)$ with equality if and only if H has no cyclic summand of order a multiple of r.

Theorem 19 *Let r be an odd prime number. Let $b \geq 2$. Then*

$$p_r\tau \in \begin{cases} \mathcal{I}_r^{b-2} & \text{if } b \text{ is even,} \\ \mathcal{I}_r^{b-3} & \text{if } b \text{ is odd.} \end{cases}$$

Furthermore, if b is odd, then $p_r\tau \pmod{\mathcal{I}_r^{b-2}} \in \mathcal{I}_r^{b-3}/\mathcal{I}_r^{b-2}$.

In the case $b \geq 3$ is odd, there is also an explicit formula expressing the triple cup product f_r in terms of $p_r\tau$, which is analogous to Theorem 18.

We now discuss the relation with the linking pairing. The linking pairing is a symmetric bilinear pairing $\lambda_M : \text{Tors } H \times \text{Tors } H \to \mathbb{Q}/\mathbb{Z}$ defined as follows. Consider the long exact sequence

$$... \to H^1(M;\mathbb{Q}) \to H^1(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Q}) \to ...$$

associated to the exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. The Bockstein homomorphism $\beta : H^1(M;\mathbb{Q}/\mathbb{Z}) \to H^2(M;\mathbb{Z})$ has image $\text{Tors } H^2(M;\mathbb{Z}) = \text{Tors } H$. Let $x, y \in \text{Tors } H$. Pick $\tilde{x} \in H^1(M;\mathbb{Q}/\mathbb{Z})$ such that $\beta(\tilde{x}) = x$. Define $\lambda_M(x,y) = (\tilde{x} \cup y)([M]) \in \mathbb{Q}/\mathbb{Z}$. A short verification shows that $\lambda_M(x,y)$ does not depend on the choice of the lift $\tilde{x}$ of x.

The linking pairing λ_M can be explicitly recovered from the torsion τ, provided that M is a rational homology 3-sphere (that is, H is finite). According to Lecture 5, the maximal abelian torsion τ can be seen as a H-equivariant map

$$\text{Eul}(M) \to \mathbb{Q}[H], \ e \mapsto \tau(M,e) = \sum_{h \in H} a^e(h)h.$$

Recall that H-equivariance means $\tau(M, h \cdot e) = h \cdot \tau(M,e)$, for any $h \in H$. Since $\mathbb{Q}[H]$ is canonically isomorphic to the ring of $\mathbb{Q}$-valued functions on H (with convolution as a product), $\tau(M,e)$ may be viewed as a map $H \to \mathbb{Q}$.

Theorem 20 (see [19]**)** *The following relation holds:*

$$\forall \ h,k \in H, \quad \tau(M, hk \cdot e) - \tau(M, h \cdot e) - \tau(M, k \cdot e) + \tau(M,e) = -\lambda_M(h,k) \bmod 1.$$

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